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THE POLES OF FINITE GROUPS OF FRACTIONAL LINEAR SUBSTITUTIONS IN THE COMPLEX PLANE.

By W. B. CARVER, Cornell University.

In his Lectures on the Icosahedron, Klein shows that any finite group of fractional linear substitutions may be transformed to a canonical form such that each substitution will correspond (by the ordinary stereographic projection) to a rotation of the sphere which has the unit circle as a great circle. In this canonic form the two poles of each substitution are what we may call *skew-inverse* with respect to the unit circle; *i. e.* they lie on a line with the center, and the product of their distances from the center is -1. Two points may be said to be *skew-inverse* with respect to any circle C of radius r if they are collinear with the center and the product of their distances from the center is $-r^2$. This relation, unlike the ordinary *inverse* relation, is not invariant under the general fractional linear substitution. Hence the following two questions (which are practically equivalent) naturally arise:

(1). If a finite group of fractional linear substitutions is not in canonic form, do its operations correspond by a stereographic projection to rotations of a sphere having some other circle of the complex plane as a great circle?

(2). If upon all the pairs of points skew-inverse with respect to some circle of the complex plane one operates with an arbitrary fractional linear substitution, will they go into pairs skew-inverse with respect to a circle; and if so, what circle?

It is the purpose of this paper to answer these questions in reverse order.

If the points of a sphere are projected upon the plane of a great circle C from a pole of C, a familiar one-to-one correspondence is established which has the following well-known properties:

Circles on the sphere correspond to circles* in the plane.

Great circles on the sphere correspond to circles in the plane which

* The term circle is to be understood to include straight lines.

cut C at the ends of a diameter of C. These circles may be said to be "diametric" to C.

The ends of a diameter of the sphere correspond to a pair of points skew-inverse with respect to C.

From the above properties of the stereographic projection one may readily deduce the following theorems:

(a). Two circles, each diametric to a circle C, intersect in a pair of points *skew-inverse* with respect to C.

(b). Any circle passing through a pair of points skew-inverse with respect to C is diametric to C.

(c). Through any two pairs of points skew-inverse with respect to C, one and only one circle can be passed.

Now let C be any circle (not a straight line) in the complex plane; and let

$$z'=\frac{\alpha z+\beta}{\gamma z+\delta}$$

be an arbitrary fractional linear substitution on the complex variable z (subject only to the restriction $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$). The point $-\frac{\delta}{\gamma}$ is sent by this substitution to infinity, and this point has a skew-inverse partner with respect to C. Of this pair of points, let P be the one inside, and Q the one outside of the circle C. Let r be the radius of C, and d the distance from the center of C to P. Then let K be the circle which is diametric to C, and with respect to which P and Q are ordinary inverse points. This circle is always uniquely determined; for it must pass through the ends of the diameter of C which is perpendicular to the line PQ, and its center is at a distance $d_1=\frac{2r^2d}{r^2-d^2}$ from the center of C. (Since $r>d$, d_1 has the same sign as d, and hence P and the center of K lie on the same side of the center of C.

If now A and B are any pair of skew-inverse points with respect to C, and the substitution $z'=\frac{\alpha z+\beta}{\gamma z+\delta}$ sends A into A', B into B', and the circle K into the circle* K', it may be very readily seen that A' and B' will be skew-inverse with respect to K'. For the two points P' and Q' will be inverse with respect to K'; and since one of them is at infinity, the other is the center of K'. The points A, B, P, and Q lie on a circle M (theorem (c)), which goes into a straight line M' through the center of K'. The line PQ goes into the line $P'Q'$. The intersections R and S of the line PQ and the circle K go into R' and S', ends of a diameter of K'. Since R and S are skew inverse with respect to C (theorem (a)), A, B, R, and S lie on a circle N which goes into a circle N' diametric to K'. A and B, intersections

* It should be noted that K' must be a proper circle and not a straight line.

of the circles M and N, go into A' and B', intersections of M' a diameter of K', and N' a circle diametric to K'. Hence A' and B' are inverse with respect to K'.

This answers question (2). And question (1) may then be answered in the affirmative in view of the two following facts:

First, any arbitrary finite group of fractional linear substitutions may be obtained by transforming one of Klein's canonical groups by a fractional linear substitution; and

Secondly, if the poles of a fractional linear substitution are skew-inverse with respect to some circle C, then this substitution corresponds, by stereographic projection, to a rotation of the sphere which has C as a great circle.

HISTORICAL NOTE ON THE NEWTON-RAPHSON METHOD OF APPROXIMATION.

By FLORIAN CAJORI, Colorado College.

Newton explained his method of approximation to the real roots of numerical equations in a tract, *De analysi per aequationes numero terminoraum infinitas,** which is celebrated chiefly as containing the first announcement of the principle of fluxions and the binomial theorem. Newton placed it in the hands of his teacher, Isaac Barrow, in 1669. Barrow sent it to John Collins, a member of the Royal Society, who, for his zeal for collecting and diffusing scientific information, received the sobriquet of "English Mersenne." The tract became known to some correspondents of Collins and to friends of Newton, but it was not printed until 1704 and 1711. Substantially the same explanation of his method of approximation to the roots of numerical equations was given by Newton in a second tract, the *Methodus fluxionum et serierum infinitarum,* which was planned for publication in 1671, but was not printed until 1736.

The earliest printed account of Newton's method of approximation appeared in Wallis' *Algebra,* London, 1685, chapter 94. Wallis explains Newton's process of solving $y^3-2y-5=0$, and derives, in the equation $y^3+axy+aay-x^3-2a^3=0$, Newton's value of y expressed in terms of a and x by a rapidly converging series. The two equations are treated by Newton in much the same manner. The solution of $y^3-2y-5=0$ is exhibited in Wallis' *Algebra,* as also in the two tracts of Newton referred to above, in paradigms which are identical except in some of the last digits in the decimal fractions. The following is copied from page 268 of Vol. I of Newton's *Opera* (Ed. Horsley):

* Isaac Newton's *Opera* (Edition by Horsley), 1779-1785, Vol. I, pp. 268-269.

$y^3-2y-5=0$		$+2.10000000$ -0.00544853 $+2.09455147=y$
$2+p=y$	$+y^3$ $-2y$ -5	$+8+12p+6p^2+p^3$ $-4-2p$ -5
	Summa	$-1+10p+6p^2+p^3$
$0{,}1+q=p$	$+p^3$ $+6p^2$ $+10p$ -1	$+0{,}001+0{,}03q+0{,}3q^2+q^3$ $+0{,}06\ \ +1{,}2\ \ +6{,}0$ $+1\ \ \ +10,$ $-1,$
	Summa	$+0{,}061+11{,}23q+6{,}3q^2+q^3$
$-0{,}0054+r=q$	$+6{,}3q^2$ $+11{,}23q$ $+0{,}061$	$+0{,}000183708-0{,}06804r+6{,}3r^2$ $-0{,}060642\ \ \ +11{,}23$ $+0{,}061$
	Summa	$+0{,}000541708+11{,}16196r+6{,}3r^2$
$-0{,}00004854+s=r$		

We see that Newton takes as the initial approximation to the real root $y=2$. He expresses himself thus: "Sit 2 numerus qui minùs quàm decimâ sui parte differt à radice quaesitâ. Tum pono $2+p=y$..." The equation in p becomes $p^3+6p^2+10p-1=0$. Neglecting the higher powers of p, he gets $10p-1=0$; taking $p=.1+q$, he gets $q^3+6.3q^2+11.23q+.061=0$. From $11.23q+.061=0$ he obtains $q=-.0054+r$, and by the same process, $r=-.00004853$. Finally, $y=2+.1-.0054-.00004853=2.09455147$.

Newton seems quite aware that his method of approximation may fail. If there is doubt, he says, whether $p=.1$ is sufficiently close to the truth, find p from $6p^2+10p-1=0$, but he does not inquire whether even this latter method will always answer. He gives an explanation of the solution of $y^3-2y-5=0$, but does not develop his method further. No other examples are given.

Wallis does not praise Newton's method over the older; he merely states that it "is very different from that of Vieta, Oughtred, and Harriot, which is commonly received." The words "very different" are in marked contrast to the statements of the modern historians, H. Hankel* and M. Cantor,† who make Vieta's method of approximation appear as almost identical with the procedure given by Newton. The two are not the same. The essential difference lies in the divisors used in finding the successive approximations. If r is the approximation already reached, then Newton uses a divisor which in our modern notation takes the form $f'(r)$; Vieta's divisor may be expressed in the form $|f(r+s_1)-f(r)|-s_1^n$, where $f(x)$ is the left side of the equation $f(x)=k$, n the degree of the equation; and s_1 is

* H. Hankel, *Geschichte der Mathematik*, Leipzig, 1874, pp. 369, 370.

† M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Bd. II, Leipzig, 1900, pp. 640, 641.

a unit of the denomination of the digit next to be found. Thus, in Vieta's example, $x^5-5x^3+500x=7905504$, if $r=20$, Vieta's divisor is 878295, while Newton's divisor is 794500. The inaccuracy of Hankel and Cantor consists in attributing to Vieta the same divisor as that used by Newton. This error probably arose from the fact that in some quadratic equations given by Vieta the two divisors happen to be the same in value. The advantage of the Newtonian divisor over that of Vieta lies in the smaller amount of computation usually called for.

The study of Newton's solution of $y^3-2y-5=0$ reveals the fact that his method of approximation is not the same as what is called "Newton's method" in modern text-books. If r is the initial approximation to a root of the equation $f(x)=0$, then $r_1=r-\frac{f(r)}{f'(r)}$ and $r_2=r_1-\frac{f(r_1)}{f'(r_1)}$, are designated in modern manuals as approximation by "Newton's method." But from the standpoint of the practical computor, this procedure is quite different from that of Newton. Newton derives, as we have seen, each successive step p, q, r of approach to the root, from a *new* equation; while in the modern process just described each approach is made by substitution in the *original* equation.

This modification of Newton's process was first effected by Joseph Raphson (1648(?)—1715(?)), a Fellow of the Royal Society, who published in 1690 in London, a booklet, bearing the title, *Analysis aequationum universalis.* A second edition, with an appendix, appeared in 1697.*

DeMorgan remarks that Raphson does not mention Newton; Raphson evidently considered the difference sufficient for his method to be classed independently. He does not mention Vieta. In our opinion, Raphson's paper is of much greater historical interest than is ordinarily attached to it. Its importance lies in the modification which it makes of Newton's method. Lagrange recognized the modified process as "plus simple que celle de Newton;" and it has now supplanted Newton's method.

In view of the facts it is doubtful whether the method of approximation described by Raphson should be named after Newton alone. In the first place, the prcesses used by Newton, though not identical to that of Vieta, resembles it. Newton merely simplified the divisor used. Hence the honor of invention falls largely on Vieta. In the second place, Newton did not develop his method further than simply to solve the cubic $y^3-2y-5=0$. That Raphson worked independently of Newton we doubt. But Raphson's version of the process has been accepted as an improvement. It would seem, therefore, that the "Newton-Raphson method" would be a designation more nearly representing the facts of history than is "Newton's method."

We proceed to the description of some details in Raphson's publica-

* A copy of the edition of 1697 is in the New York Public Library. As this copy is in the reference department, I was not able to borrow it through a library loan. My knowledge of Raphson's book is derived mainly from the extracts given in the Latin edition of Wallis' *Algebra*, London, 1693, and the extracts given in an appendix, by Maseres, to W. Frend's *Principles of Algebra*, London, 1796, pp. 467, 468.

tion. We have pointed out that the form now so familiar, $\frac{f(r)}{f'(r)}$, was not used by Newton, but was used by Raphson. To be sure, Raphson does not use this notation; he writes $f(r)$ and $f'(r)$ out in full as polynomials. Nor does Raphson find the first derivative by the rule of the calculus; his operations are purely algebraical. He gives "canons" or "theorems" for the forms of the fractions, in which he writes down the polynomials $f(r)$ and $f'(r)$ for each equation up to that of the tenth degree inclusive. From the standpoint of algebraic notation it may be of interest to quote his expressions for one of these, say the quintic. Taking 1, b, c, d, ... as the coefficients of the given equation and g as the value of the approximation already reached, he writes for the quintic the following "canon:"

"*Pro potestate Quinta*

$$\begin{array}{rr} ggggg & \left.\begin{array}{r} 5gggg \\ 4bggg \\ 3cgg \\ 2dg \\ f \end{array}\right\} x\text{''} \\ bgggg & \\ cggg & \\ dgg & \\ fg & \end{array}$$

The terms in the left hand column represent $f(g)$, those in the right hand column $f'(g)x$. With admirable consistency, Raphson gives the "canon," "pro potestate decima," by writing out each of the ten factors in g^{10}.

Newton and Raphson explained their methods only in connection with rational integral algebraic equations. The extension of the Newton-Raphson method to irrational and transcendental equations appears to have been made for the first time by Thomas Simpson, in his *Essays...on Mathematics*, London, 1740, p. 81. He does not mention Newton and Raphson, and calls his procedure a "new method."

Nearly all eighteenth century writers and most of the early writers of the nineteenth century carefully discriminate between the method of Newton and that of Raphson. Then appear writers like Euler, Laplace, Lacroix and Legendre, who explain the Newton-Raphson process, but use no names. Finally, in a publication of Budan in 1807,* in those of Fourier of 1818 and 1831,† in that of Dandelin in 1826,‡ the Newton-Raphson method is attributed to Newton. The immense popularity of Fourier's writings led to the universal adoption of the misnomer "Newton's method" for the Newton-Raphson process.

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* *Nouvelle méthode pour la résolution d. équat. num.*, Paris, 1807, p. 78.

† *Bulletin d. sciences par la société philomatique de Paris*, année 1818, p. 61; Fourier, *Analyse des équations déterminées*, 1831, pp. 169, 173, 177, etc.

‡ *N. mémoires d. l' académie r. d. sciences et belles-lettres de Bruxelles*, T. 3, 1826, pp. 28, 29.

ON CERTAIN SPACE GENERALIZATIONS.

By REV. ALAN SPENCER HAWKESWORTH, University of Pittsburgh.

By its essential concept, a rectangular figure of the nth dimension must be ultimately bounded by $2n$ figures of the next lower, $(n-1)$st dimension; must have 2^n vertices; and from every such vertex n mutually perpendicular lines must spring. On the other hand, each such line has two bounding points, and hence the figure must have $\frac{1}{2}n$ times as many edges as points, or a total of $n(2^{n-1})$.

In like manner, since from every edge starts a square of four edges in two dimensional space, from every edge two squares in three dimensions, three in four dimensions, and $n-1$ squares in n dimensions; it follows that the number of said squares in our n dimensional rectangle must be

$$\frac{(n)\ (n-1)}{2!}2^{n-2}.$$

By similar reasoning, the number of bounding cubes must be

$$\frac{(n)\,(n-1)\,(n-2)}{3!}2^{n-3}.$$

Hence, to sum up, the a-dimensional boundaries of any rectangular figure in n dimensions, must be $\frac{n!}{(n-a)!\,(a)!}.2^{n-a}$.

Similar reasoning shows that figures in n dimensions, analogous to the equilateral triangle, and the tetrahedron, will be bounded by $n+1$ figures of the next lower $(n-1)$st dimension; and have $n+1$ vertices, $\frac{(n+1)\,(n)}{2!}$ edges, $\frac{(n+1)\,(n)\,(n-1)}{3!}$ triangular faces, $\frac{(n+1)\,(n)\,(n-1)\,(n-2)}{4!}$ bounding tetrahedrons; and in brief, $\frac{(n+1)!}{(n-a)!\ (a+1)!}$ boundaries of the ath dimension.

From which it follows that the numerical values of the boundaries must rise and fall symmetrically to a medial value; the number $n+1$ of the vertices equalling that of the last $n-1$ boundaries, as stated; the edges and the $n-2$ boundaries each being $\frac{(n+1)\ (n)}{2!}$; the planes and the $n-3$ boundaries each being $\frac{(n+1)\,(n)\,(n-1)}{3!}$, and so on.

Note that the formulae hold true, even when the apparently absurd

question is asked concerning the number of higher dimensional boundaries in a lower dimensional figure, that is, when a is taken greater than n. For the negative and fractional, or reciprocal, result, which we then obtain, as for example, that there are $-\frac{1}{6}$ cubes in a square, means simply that, reversely and reciprocally, there are $+6$ squares in every cube.

Lastly, in extension of Euler's theorem that in all three-dimensional rectilinear solids the sum of their vertices and surfaces is equal to their edges plus 2, we can deduce this general formula, namely: In every even $(2n)$-dimensional rectilinear figure the algebraic sum of its boundaries, taken in their sequence alternately plus and minus, is always zero, while the similar summation of those of every odd $(2n+1)$-dimensional figure is always $+2$.

The proof of this is as follows: First, for rectangular and tetrahedroidal figures.

Any rectangular figure of the nth dimension can be generated by the rectangular movement in n dimensional space of the corresponding figure in the next lower $(n-1)$st dimension. Through this movement the vertices, edges, squares, and cubes, etc., of the generating figure must be given in duplicate; while each vertex must trace an extra edge, each edge a square, each square a cube, and so on. So that letting v, e, p, s, etc., stand for the vertices, edges, planes, surfaces, etc., of our generating $(n-1)$st dimensional figure; and v_1, e_1, p_1, s_1, etc., for those of the generated n dimensional figure, we have always $v_1=2v$; $e_1=2e+v$; $p_1=2p+e$; $s_1=2s+p$; etc.

Now let us represent by l the number and character of the ultimate $(n-2)$ boundaries of our $(n-1)$ figure, which figure we will also call an m. When the last two boundaries of our generated (n) figure must plainly be

$$l_1=(2l+\dots) \text{ and } m=(2m+l).$$

The algebraic sum of the boundaries of our m figure of $(n-1)$ dimensions, with alternate terms taken $+$ and $-$, will be $+v-e+p-s+\dots\dots\mp l$, where the sign of the last term l will be $-$ or $+$ according as m is an even or an odd dimensional figure.

Similarly then the sum of the boundaries, taken alternately $+$ and $-$, of our generated n figure will be $+v_1-e_1+p_1-s_1+\dots\mp l_1\pm m_1=+2v-(2e+v)+(2p+e)-(2s+p)+\dots\mp(2l+\dots)\pm(2m+l)=+v-e+p-s+\dots\mp l\pm 2m$, the sign of the last term $2m$ being again $+$, when the n figure is an odd dimension, but $-$, when it is of an even dimension.

Therefore in any two successive rectangular figures of the $(n-1)$st and nth dimension respectively, the algebraic sum, as above, of the successive boundaries of n, taken alternately $+$ and $-$, will be always greater than those of $(n-1)$, by $+2$, if $(n-1)$ be even and n odd. But if conversely, $(n-1)$ is odd and n even, then the said boundaries of n are always less by -2 than those of $(n-1)$.

Beginning now with the even dimensional square, $+v-e$ is zero, and

in the odd dimensional cube $+v-e+p$ is $+2$; which we know to be correct. Hence the hypothetical fourth, sixth, eighth, etc., dimensional rectangles must always have this summation of their boundaries equal to zero, while in the case of the fifth, seventh, ninth, etc., dimensional rectangles this sum is $+2$.

Taking up now the higher dimensional figures, analogous to the equilateral triangle and the regular tetrahedron, again let $+v-e+p-s+\ldots\mp l$ be the summation of the vertices, edges, planes, solids, etc., up to the limiting l boundary of such an $(n-1)$st dimensional figure, which we will, as before, call m, the sign of the final boundary being once more $-l$, when m is an even dimensional figure, but $+l$ when it is odd; while $+v_1-e_1+p_1-s_1+\ldots\mp l_1\pm m_1$ will be, as before, the summation of the boundaries of the next higher n figure, the signs being $-l_1+m_1$ when n is of an odd dimension, but $+l_1-m_1$ when n is even.

Then the law is $v_1=v+1$; $e_1=e+v$; $p_1=p+e$; $s_1=s+p$; $\ldots m_1=m+l$. Hence $+v_1-e_1+p_1-s_1+\ldots\mp l_1\pm m_1=+(v+1)-(e+v)+(p+e)-(s+p)+\ldots\mp(l+\ldots)\pm(m+l)=+1\pm m=+2$ or zero.

So that here also the summation of the boundaries, taken alternately $+$ and $-$, is always equal to zero, when n is an even and $(n-1)$ an odd dimensional figure, but is equal to $+2$ when n is odd and $(n-1)$ even.

This then fully proves the theorem for at least rectangular and tetrahedroidal figures of the nth dimension. And that it is equally valid for any rectilinear figure whatsoever in n dimensions can also be shown.

For alike in cuboid, tetrahedroid, and the analogue of the tetrakaidecagon in the fourth dimension, from any corner must stretch four edges, six planes and four solids. So that, truncating a corner in any one of the figures, we will add four new vertices and lose an old one; or will add $+3$ vertices in all. We will create six new edges and three new planes but no new solids. hence our new figure adds $+3v-6e+3p=0$ and the previous summation of the untruncated figure is not affected,

But a still more general proof, valid in all dimensions is the following: By the dimensional concept, from every one-dimensional line must stretch two plane faces and one solid and from each plane face a solid, in all three dimensional figures. In those of the fourth dimension, every edge must touch three plane faces and three solids. While every face must bound and be common to two of the bounding solids. And thus in any n-dimensional figure each line must touch $(n-1)$ plane faces. Each face must touch $(n-2)$ solids and every solid must have $(n-3)$ fourth dimensional boundaries and so on, until at last each $(n-2)$ boundary must touch $(n-[n-2])=+2$ of the $(n-1)$ boundaries of our n-dimensional figure.

Therefore taking, say, any fourth-dimensional figure, let us build it up step by step from its component l solids. And let us represent the vertices, edges and planes of our first solid by $+v_1-e_1+p_1$; those of the second by $+v_2-e_2+p_2$, and so on.

Take the first solid $+v_1-e_1+p_1=+2$, and join it to the second solid, so that an identical face in each coincides. Now were we dealing with three-dimensional space, we should evidently have lost two plane faces, each having $v=e$; the said faces passing into the interior of our new composite solid figure, which solid figure has yet $+v-e+p=+2$.

But we are concerned with four-dimensional space, wherein all the planes, etc., of our first solid remain unaltered, and where the second and added solid loses but *one* face, with its equal vertices and edges, by coalescing with the similar face in the first solid. Hence now $v_2-e_2+p_2=+3$.

Similarly, adding the third bounding solid, in place of four planes disappearing, as they would in three-dimensional space, we lose by coalescing but two, and thus $+v_3-e_3+p_3=+4$.

And so on, until we come to the last solid but one, $(l-1)$. When as before, $+v_{l-1}-e_{l-1}+p_{l-1}=+l$.

But upon adding the last l bounding solid, no new boundaries appear. All of its vertices, edges, and planes coalescing with those already existing in our built up figure, so that $+v_l-e_l+p_l=+l=$the solid boundaries, by definition. And hence $+v-e+p-s=0$, for any and all fourth-dimensional figures.

And the same would be true, were a fifth-dimensional figure given, with m bounding fourth-dimensional figures. For, taking the first of the said m figures, we have $+v_1-e_1+p_1-s_1=0$. Adding the second, and thereby losing but one solid, in place of two, we have $+v_2-e_2+p_2-s_2=-1$. The third figure, in like manner, gives us $+v_3-e_3+p_3-s_3=-2$. Until finally, $+v_{m-1}-e_{m-1}+p_{m-1}-s_{m-1}=-(m-2)=+2-m=+v_m-e_m+p_m-s_m$. Or $+v-e+p-s+m=+2$.

And quite similarly, for any dimension. The odd dimensions always adding $+2$ to the zero summation of the previous even figure, and conversely, the even always taking -2 from the $+2$ summation of the previous odd, as we similarly proved for rectangular and tetrahedroidal figures.

Lastly, were we to count the n-dimensional figure itself as its own boundary, and thus add -1 or $+1$ to our previous summation, according as n is odd or even, then we obtain the still more simple rule that any and all such summations always give $+1$, whatever may be the rectilinear figure.

In this case the binomial development of $(2-1)^n$ will give us both the sequence and the summation of the boundaries of a rectangular n-dimensional figure; while, in a similar way, the symmetrical sequence and summation of those of an n-dimensional tetrahedroidal figure can be represented by $-(1-1)^{n+1}$, omitting the first term of the binomial development.

NOTE ON CAUCHY'S INTEGRAL TEST.

By M. B. PORTER, University of Texas.

It does not seem to have been generally noticed that series which consist of a monotonic decreasing sequence of terms of the same sign lend themselves to calculation as readily as those whose signs alternate provided the integral $\int_n^\infty u_n dn$ is sufficiently simple. We have at once, if the terms are all positive,

$$\int_n^\infty u_n dn \leqq u_n + u_{n+1} + u_{n+2} + \ldots \text{etc.} \leqq u_n + \int_n^\infty u_n dn.$$

Thus if to the sum of the first $n-1$ terms we add $\frac{u_n}{2} + \int_n^\infty u_n dn$, our approximation will not differ from the sum of the series by more than $u_n/2$, that is, by less than half of the first term we omit.

Thus to calculate the series $\sum_1^\infty n^{-2}$ the correction after $n-1$ terms is $\frac{1}{2n^2} + \frac{1}{n}$, and the calculated value is in error not more than $\frac{1}{2n^2}$. This fact can be used to calculate with a minimum of labor such slowly convergent series as those represented by the logarithmic scales of DeMorgan.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

346. Proposed by E. B. ESCOTT, Professor of Mathematics, University of Michigan.

Solve completely by quadratics:

$$\frac{2x}{1-x^2} = y; \quad \frac{2y}{1-y^2} = z; \quad \frac{2z}{1-z^2} = w; \quad \frac{2w}{1-w^2} = x.$$

Solution by the PROPOSER.

Let $x = \tan\phi$, then $y = \tan 2\phi$, $z = \tan 4\phi$, $w = \tan 8\phi$, $x = \tan 16\phi = \tan\phi$, $\tan 16\phi - \tan\phi = 0$, or $\frac{\sin 15\phi}{\cos 16\phi . \cos\phi} = 0$.

If $\cos\phi=\infty$, $\tan\phi=\pm i$.

If $\sin 15\phi=0$, $\phi=\frac{n\pi}{15}$ $(n=0, 1, 2, \ldots, 14)$.

$\therefore x=\pm i, 0, \tan\frac{\pi}{15}, \tan\frac{2\pi}{15}, \ldots, \tan\frac{14\pi}{15},$

$$=\pm i,\ 0,\ \pm\frac{\sqrt{[10+2\sqrt{(5)}]}-\sqrt{(15)}+\sqrt{(3)}}{\sqrt{[30+6\sqrt{(5)}]}+\sqrt{(5)}-1},\ \pm\frac{\sqrt{(15)}+\sqrt{(3)}-\sqrt{[10-2\sqrt{(5)}]}}{\sqrt{[30-6\sqrt{(5)}]}+\sqrt{(5)}+1},$$

$$\pm\frac{\sqrt{[10-2\sqrt{(5)}]}}{\sqrt{(5)}+1},\ \pm\frac{\sqrt{[10+2\sqrt{(5)}]}+\sqrt{(15)}-\sqrt{(3)}}{\sqrt{[30+6\sqrt{(5)}]}-\sqrt{(5)}+1},\ \pm\sqrt{(3)},$$

$$\pm\frac{\sqrt{[10+2\sqrt{(5)}]}}{\sqrt{(5)}-1},\ \pm\frac{\sqrt{(15)}+\sqrt{(3)}+\sqrt{[10-2\sqrt{(5)}]}}{\sqrt{[30-6\sqrt{(5)}]}-\sqrt{(5)}-1}.$$

(These values are taken from Hobson's *Plane Trigonometry.*)

Professor J. Scheffer gave a partial solution, following the method in the above solution. Mr. Spuner sent in a similar solution, with results given in decimals instead of radicals.

347. Proposed by GUSTAVE JACOBSON, A. M., Public Accountant, Chicago, Ill.

A corporation needing some additional capital for a short term of years, issues \$300,000 of debenture bonds carrying 6% interest, and payable 1/5 each year for 5 years. Coupons are attached to the bonds maturing every six months; the bonds are sold at 90 flat. What average rate of interest does the company pay for the money, including interest on interest?

Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let $2x$=the rate of interest paid per annum, payable semi-annually; then x=the rate of semi-annual interest, compound. Only \$270,000 was used by the corporation, which was repaid, interest in 10 payments and principal in 5 payments; and they alternate, the first each six months and the second at the end of each year as follows: (1), \$9,000; (2), \$69,000, (3), \$7,200; (4), \$67,200; (5), \$5,400; (6), \$65,400; (7), \$3,600; (8), \$63,600; (9), \$1,800, and (10), \$61,800.

The problem may now be treated, algebraically, as a question in partial payments under the U. S. Court Rule; and we have, after taking out the factor 600, arranging, dropping the dollar sign as follows:

$$450(1+x)^{10}-[15(1+x)^9+115(1+x)^8+12(1+x)^7+112(1+x)^6$$
$$+9(1+x)^5+109(1+x)^4+6(1+x)^3+106(1+x)^2+3(1+x)]=103.$$

Observe that as arranged the first member has 2 terms, and their difference

$=103$, As the debenture was discounted, the rate realized on the investment must be greater than 3% as a semi-annual rate. We try 5% and find it too small; we then try 5.0275%, with $(1+x)=1.050275$, and find the first member$=102.9337$, which is too small; we then try $(1+x)=1.050285$ and obtain the value 102.9531, still too small. By double position we have a corrected value and $(1+x)=1.050309$ and obtain 103.0304, which is now too large. By double positon we have $(1+x)=1.050291$, and obtain 102.9937, again too small. By double position again we have $(1+x)=1.0502941$ and obtain 103.0033, now too large. We now again use double position and obtain, $(1+x)=1.05029269$, this will give a value which will differ by less than 1% of a mill. We now have for a good approximation a semi-annual rate, 5.029269%; or the rate per annum payable semi-annually, 10.058538%.

Also solved by S. A. Corey who got for a result 10.28 per cent.; A. H. Holmes, whose result was 10.07 per cent., and the proposer, whose result was 10.0532 per cent. These contributors used practically the same method of solution. Mr. Spuner also sent in an excellent solution of 345, which reached us too late for credit in the January number.

CALCULUS.

298. **Proposed by C. N. SCHMALL, New York City.**

Prove, by calculus, that if two regular polygons have equal perimeters, that which has the greater number of sides has the greater area.

Solution by E. L. SHERWOOD, Shady Side Academy, Pittsburg, Penn., and the PROPOSER.

In any regular polygon, let $2p$=the perimeter, n=number of sides.

$\therefore \dfrac{2p}{n}$=each side, and $\dfrac{p}{n\tan(\pi/n)}$=the apothem.

$$\frac{p^2}{n\tan(\pi/n)}=\frac{p^2}{n}\cot\frac{\pi}{n}=\text{area}\ldots(1).$$

Put $u=n\tan\dfrac{\pi}{n}\ldots(2)$; hence the area will be a maximum when u is a minimum.

From (2) we have

$$\frac{du}{dn}=\tan\frac{\pi}{n}-\frac{\pi}{n}\sec^2\frac{\pi}{n}=\tan\phi-\phi\sec^2\phi \text{ (where } \frac{\pi}{n}=\phi)=\sin\phi\sec\phi-\phi\sec^2\phi$$

$$=\tfrac{1}{2}(2\sin\phi\sec\phi-2\phi\sec^2\phi)=\tfrac{1}{2}\sec^2\phi\left(\frac{2\sin\phi}{\sec\phi}-2\phi\right)=\tfrac{1}{2}\sec^2\phi(\sin2\phi-2\phi),$$

$$=\tfrac{1}{2}\sec^2\frac{\pi}{n}\left(\sin\frac{2\pi}{n}-\frac{2\pi}{n}\right).$$

This expression is negative for $n>2$ and therefore decreases as n increases. Hence the area increases as n increases. Q. E. D.

NOTE. Equating the derivative to zero, we have

$$\tan\frac{2\pi}{n}-\frac{2\pi}{n}=0.$$

This equation is true only for very small values of the angle. In fact, it approximates to exactitude as the angle $2\pi/n$ approaches zero. Hence, the area of the polygon opproaches its maximum value (as a limit) as n approaches ∞; it reaches its maximum when $n=\infty$. In that case the polygon becomes a circle, which is, therefore, the greatest of all isoperimetric plane figures. In a similar manner, it may be shown that of all plane figures of given area the circle has the least periphery.

Also solved by J. E. Sanders, V. M. Spunar, J. Scheffer, and S. G. Barton.

299. Proposed by JOSEPH V. COLLINS, Ph. D., State Normal School, Stevens Point, Wisconsin.

A cow is pasturing outside a circular field containing 10 acres. What length of rope will allow her to graze over exactly two acres?

Solution by J. SCHEFFER, A. M., Hagerstown, Md.; J. E. SANDERS, Columbus, Ohio, and S. G. BARTON, Ph. D., Clarkson School of Technology.

It is assumed that the cow is tethered at a point in the circumference of the field.

It is evident that the area of the pasture consists of a semi-circle, of radius equal to the length of the rope and two portions contained between the circular boundary of the field and the involute of the circle.

Let a be the radius of the field, l the length of the rope. The intrinsic equation of the involute of a circle is

$$s=a\frac{\psi^2}{2}.$$

Considering the element of area contained between this involute and its evolute, that is, the circle, and two radii of curvature, we have

$$dA=\rho\frac{ds}{2},\quad \rho=\frac{ds}{d\psi}=a\psi;\quad \therefore s=\frac{\rho^2}{2a},\quad ds=\frac{\rho}{a}d\rho.\quad \therefore dA=\frac{\rho^2}{2a}d\rho.$$

The limits of ρ for area involved are evidently 0 and l.

$$\therefore A=\int_0^l \frac{\rho^2}{2a}d\rho=\frac{l^3}{6a}.\quad \text{Hence area}=2\text{ acres}=\frac{\pi l^2}{2}+\frac{l^3}{3a}.$$

Using data of problem, $a=372.4$ feet. $.0009l^3+1.5708l^2=87120$. $l=221.9$ feet.

Solved in a slightly different way by V. M. Spunar.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

Edited by Dr. G. E. Wahlin, University of Illinois.

175. Proposed by H. C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Nebraska.

Show (a) that $[(2n)!]/[(n+1)!n!]$ is an integer, and (b) that $[(2a)!(2b)!]/[a!b!(a+b)!]$ is an integer.

Solution by G. E. WAHLIN, Ph. D., University of Illinois.

(a) To show that $\dfrac{(2n)!}{(n+1)!(n!)^2}$ is an integer.

$$\frac{(2n)!}{(n+1)\ (n!)^2}=\frac{(2n)!}{(n+1)!\,n!}=\frac{2n(2n-1)\ (2n-2)\dots(n+1)}{(n+1)!}.$$

$I=\dfrac{2n\ (2n-1)\dots(n+1)}{n!}$ is an integer, being the quotient of n successive integers by $n!$.

For the same reason, $I'=\dfrac{(2n+1)\ (2n)\ (2n-1)\dots(n+1)}{(n+1)!}$ is an integer.

But $I'=\dfrac{2n+1}{n+1}I$; and since $2(n+1)-(2n+1)=1$,

the numerator and denominator are relatively prime in $\dfrac{2n+1}{n+1}$, and therefore since $I.\dfrac{2n+1}{n+1}$ is an integer, I must be divisible by $n+1$, or $\dfrac{I}{n+1}$ is an integer.

But $\dfrac{I}{n+1}=\dfrac{2n\ (2n-1)\dots(n+1)}{(n+1)!}=\dfrac{2n!}{(n+1)\ (n\,!)^2}$.

(b) To show that $\dfrac{(2a)!(2b)!}{a!b!\ (a+b)!}$ is an integer.

It is evident that the given expression is equal to

$$\frac{(a+1)\ (a+2)\dots.2a(b+1)\ (b+2)\dots.2b}{(a+b)!}\dots(1),$$

and it is therefore sufficient to show that this is an integer.

Let p be any prime less than or equal to $a+b$, and n any integer such than $p^n \leqq a+b$. Moreover, let $a=kp^n+c$ and $b=k'p^n+c'$, where c and c' are both less than p^n.

We shall determine the number of multiples of p^n in the sequence

$$a+1,\ a+2,\ \ldots,\ 2a \ldots (2).$$

Consider first the case when $c=p^n-1$. Then $a+1=(k+1)p^n$ and $2a=2kp^n+2p^n-2=(2k+1)p^n+p^n-2$, and hence $(k+1)p^n$ is, in this case, the first multiple of p^n in the sequence, and $(2k+1)p^n$ is the last. If we let v_n be the number of such multiples in the sequence, since they form an arithmetic progression, we have

$$(2k+1)p^n=(k+1)p^n+(v_n-1)p^n.$$

Solving for v_n, we get $v_n=k+1$.

If $\frac{p^n-1}{2}<c<p^n-1$, then $a+1=kp^n+c+1$, and $2a=2kp^n+2c=(2k+1)p^n+d$, $d<p^n$. Therefore, in this case again, the first and last multiples of p^n in (2) are, respectively, $(k+1)p^n$ and $(2k+1)p^n$. As before, therefore, $v_n=k+1$.

If $c \leqq \frac{p^n-1}{2}$, then $a+1=kp^n+c+1$, and $2a=2kp^n+2c$, $2c<p^n$, and in this case the first and last multiples of p^n in the sequence are, respectively, $(k+1)p^n$ and $2kp^n$. As above, we then find that $v_n=k$. We have, therefore, that

$$v_n=k+1 \text{ when } \frac{p^n-1}{2}<c \leqq p^n-1; \quad v_n=k \text{ when } c<\frac{p^n-1}{2}.$$

In the same way, letting v_n' be the number of multiples of p^n in the sequence, $(b+1)$, $(b+2)$, ..., $2b$, we find that

$$v_n'=k'+1 \text{ when } \frac{p^n-1}{2}<c' \leqq p^n-1; \quad v_n'=k' \text{ when } c' \leqq \frac{p^n-1}{2}.$$

If we now let m be the largest integral exponent for which $p^m \leqq a+b$, since all multiples of p^n are also multiples of the lower powers of p, it is not difficult to see that $v_1+v_2+\ldots+v_m$ is the power to which p enters as a factor in the product $(a+1)(a+2)$, ..., $2a$, and $v'_1+v'_2+\ldots+v'_m$ is the expo-

nent of p in the product $(b+1)(b+2)$, ..., $2b$. Hence the power of p in the numerator of (1) is

$$s=v_1+v_2+\ldots+v_m+v_1'+v_2'+\ldots v_m'.$$

We shall next turn our attention to the sequence 1, 2, 3, ..., $a+b$. In this the first multiple of p^n is p^n and as $a+b=(k+k')p^n+c+c'$ the last is $(k+k'+1)p^n$ or $(k+k')p^n$, according as $c+c'$ is greater than or less than p^n, and if we let v_n'' be the number of such multiples in this sequence we have

$$v_n''=k+k'+1 \text{ when } c+c'>p^n; \quad v_n''=k+k' \text{ when } c+c'<p^n.$$

Hence, as before, the exponent of p in the product $(a+b)!$ is

$$t=v_1''+v_2''+\ldots+v_m''.$$

But for $c+c'$ to be greater than p^n, at least one of the quantities c and c' must be greater than $\frac{p^n-1}{2}$, and it is therefore easily seen that $v_n''\leq v_n+v_n'$ and hence $t\leq s$.

From the condition imposed on p, this may be any prime factor of $(a+b)!$, and hence the numerator of (1) contains this prime with an exponent not less than its exponent in the denominator, and (1) is therefore an integer.

A proof of the above theorem is also found in *Die Elemente der Zahlen Theorie*, p. 37, by Bachmann.

178. **Proposed by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.**

Find a formula which gives all the integral solutions prime to 5 of the congruence $y^2+z^2\equiv 0 \pmod{5^4}$.

Solution by the PROPOSER.

Since $y^2\equiv 4z^2 \pmod 5$, $y\equiv \pm 2z \pmod 5$. Hence $y=5r\pm 2z$. Then

$$y^2+z^2=5(5r^2\pm 4rz+z^2)\equiv 0 \pmod{5^4},$$
$$5r^2\pm 4rz+z^2\equiv 0 \pmod{5^3}.$$

Since z is prime to 5, the latter gives $\pm 4r+z\equiv 0 \pmod 5$, whence $r\equiv \pm z \pmod 5$. Thus $r=5s\pm z$. Substituting in $5r^2$, ..., $\equiv 0$, we get $10z^2\pm 70sz$

$\equiv 0 \pmod{5^3}$. Hence $z \pm 7s \equiv 0 \pmod{5^2}$, so that $z=25t+7s$. Now $y=5r \pm 2z=25s \pm 7z=\pm 175t-24s$. The only solutions prime to 5 are thus

$$y=\pm 175t-24s, \quad z=25t \mp 7s.$$

Conversely, these values satisfy the given congruence.

☞Problems and solutions for this department should be sent to Dr. Wahlin, Urbana, Ill.

PROBLEMS FOR SOLUTION.

ALGEBRA.

351. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve, $y^2+yz+z^2=a^2 \ldots (1)$.
$z^2+zx+x^2=b^2 \ldots (2)$.
$x^2+xy+y^2=c^2 \ldots (3)$.

352. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve the equations, $x^3=-8y+24 \ldots (1)$.
$y^3=-8x+24 \ldots (2)$.

GEOMETRY.

381. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the number of diagonals of a *complete* polygon of n sides.

382. Proposed by PROF. R. C. ARCHIBALD, Brown Uniuersity, Providence, R. I.

Between the side of a given rhombus and its adjacent side produced, to insert a straight line of a given length and directed to the opposite corner. "Euclidean constructions" are particularly desired.

383. Proposed by S. A. COREY, Hitsman, Iowa.

Let $ABCDE$ be a pentagon, plane, or gauche, with sides AB, BC, CD- and DE, of lengths, w, x, y, and z, respectively. Construct four other pen, tagons, $AB_1C_1D_1E_1$, $AB_{11}C_{11}D_{11}E_{11}$, $AB_{111}C_{111}D_{111}E_{111}$, and $AB_{iv}C_{iv}D_{iv}E_{iv}$, having a common vertex at A, and with four consecutive sides in each parallel to the corresponding consecutive sides, AB, BC, CD, and DE, in $ABCDE$. Further, let the lengths of the sides AB_1, B_1C_1, C_1D_1, D_1E_1, AB_{11}, $B_{11}C_{11}$, $C_{11}D_{11}$, $D_{11}E_{11}$, AB_{111}, $B_{111}C_{111}$, $C_{111}D_{111}$, $D_{111}E_{111}$, AB_{iv}, $B_{iv}C_{iv}$, $C_{iv}D_{iv}$, $D_{iy}E_{iv}$, be $-wW$, xX, yY, zZ, wX, xW, yZ, $-zY$, wY, yW, zX, $-xZ$, wZ, zW, xY, and $-yX$, respectively; the minus sign indicating the reversal of direction of the corresponding side. Prove that $(W^2+X^2+Y^2+Z^2)(w^2+x^2+y^2+z^2)=E_1A_1^2+E_{11}A_{11}^2+E_{111}^2A_{111}^2+E_{iv}A_{iv}^2$.

CALCULUS.

306. Proposed by FRANCIS RUST, C. E., Pittsburg, Pa.

Express in elliptic integrals: $A_\theta=\int_0^\theta \frac{dx}{\sqrt{(1-x^4)}}$; $0<\theta<0$.

307. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology.

The maximum value is not necessarily the greatest value of a function. Show why we take the maximum value as the greatest value in practical problems in maxima and minima. Likewise minimum for least value.

MECHANICS.

259. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A uniform beam of the weight W, rests on a horizontal plane, and leans against a vertical wall, but so as *not* to lie in a vertical plane. Denoting the pressure upon the horizontal and vertical planes, respectively, by x and y, the coefficients of friction respectively, by μ and μ'; the angle which the perpendiculars from the foot of the beam upon the intersection of both planes makes with the beam by ϕ; the angle which this perpendicular makes with the direction of the friction peg by ξ; and the angle, which the projection of the beam upon the vertical wall makes with the vertical line, by ψ. To prove: $(1+\mu^2)\mu'^2\sin^4\psi-[1+\mu'^2+4\mu^2\mu'^2]\sin^2\psi+4\mu^2\mu'^2=0$; $\tan\xi=\mu'\cos\psi$, $\tan\phi=\mu'\cot\psi$, $x=\frac{\mu\cos\xi}{1+\mu\mu'\sin\psi\cos\xi}W$, $y=\frac{1}{1+\mu\mu'\sin\psi\cos\xi}W$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

181. Proposed by V. M. SPURAR, M. and E. E., Chicago, Ill.

If $2n+1$ is an odd prime p, $(2n)!\equiv(-1)^n2^{4n}(n!)^2 \pmod{p^2}$.

182. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Find two general solutions in integers of the equation $x^2=616318177y-1$.

148. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find all the multiply perfect numbers of n different prime factors and of multiplicity $n-1$.

153. Proposed by LLOYD HOLSINGER, A. B., 227 Fredonia Avenue, Peoria, Ill.

If we represent by (k, l) the greatest common divisor of k and l, and by $\phi(k)$ the number of integers prime to k and not greater than k, we have

$$\begin{vmatrix} (1,1) & (1,2) & (1,3) & \dots & (1,n) \\ (2,1) & (2,2) & (2,3) & \dots & (2,n) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (n,1) & (n,2) & (n,3) & \dots & (n,n) \end{vmatrix} = \phi(1).\phi(2).\phi(3)\dots.\phi(n).$$

157. Proposed by A. H. HOLMES, Brunswick, Maine.

Find integral values for m and n in $64m^2n^2(m^2-n^2)^2+(m^2+n^2)^4=\square$.

161. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find a solution of $x^4-5x^2+4\equiv 0 \pmod{p.\overline{2p+1}}$, where both p and $2p+1$ are odd primes.

162. Proposed by L. E. DICKSON, Ph. D., Associate Professor of Mathematics, The University of Chicago.

If p is an odd prime, find the number of incongruent integers x for which x^4+2ex^2+f is a quadratic residue of p.

166. Proposed by H. S. VANDIVER, Bala, Pa.

Eliminate any five of the seven quantities h, j, k, l, m, r, s, from:

$$j+m+r=k+l+s,$$
$$h=3(r+s)-(n+1),$$
$$j+m+2r+s=n,$$
$$hr+j^2+km+ms=jk+mr+s^2+rs,$$
$$ks+jl+rs+r^2=sh+jk+l^2+rk,$$
$$hr+jl+ks+ls+km=jm+mr+2rs+rj.$$

167. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Prove that $\Pi\dfrac{p+(-1)^{\frac{1}{2}(p+1)}}{p-(-1)^{\frac{1}{2}(p+1)}}=2$, the consecutive values of p being the natural odd primes in order.

NOTES AND NEWS.

At Princeton University Dr. G. D. Birkhoff, assistant professor, and Dr. William Gillespie, preceptor, have been promoted to full professorships in mathematics. S.

Dr. Anna J. Pell has been elected to an instructorship in mathematics at Mount Holyoke College for the year 1911-1912. She is at present conducting courses at Armour Institute, Chicago, during the illness of her husband, Professor Alexander Pell. S.

Professor E. R. Hedrick, of the University of Missouri, has leave of absence for eight months and will spend the time in travel and study abroad. S.

Professor E. B. Van Vleck, who has been on leave of absence from the University of Minnesota since June, 1910, has returned to resume his duties for the second semester. S.

Dr. H. B. Smith has been appointed Instructor of Mathematics at the University of Pennsylvania, for the ensuing term, to fill the vacancy caused by the temporary absence of Professor Evans. C.

The Department of Superintendents of the National Education Association met at Mobile, Alabama, on February 23, 24, 25, 1911. The annual meeting of the Association is to be held in San Francisco early in July. S.

Professor Gilbert A. Bliss, of the University of Chicago, who during the autumn of 1910 was traveling in Japan and other eastern countries, is now in Paris. He will return to the University of Chicago in Aprill, 1911, having completed the journey around the world. S.

Teachers' College, Columbia University, New York City, has recently issued a bibliography of mathematical works suitable for high school and normal school libraries. This is prepared by Professor David Eugene Smith and Professor Clifford Brewster Upton. Teachers who may care for this bibliography may obtain it gratis by writing to the Secretary of Teachers' College, Columbia University. S.

The first of a series of articles, later to be issued in book form, by Dr. George Bruce Halsted, appeared in February's *Open Court* under the title: The Prehuman Contributions to Arithmetic. Dr. Halsted will, in these articles, transmute the dry bones of arithmetic into a fascinating serial. A commendatory review of the first of this series of articles is given in the *Chicago Tribune* of February 9. F.

The Chicago Section of the American Mathematical Society has completed its fourteenth year, having held since its organization in 1897, twenty-eight regular meetings, at which four hundred and eighty-one papers have been presented, or an average of about eighteen papers per meeting. The first conference on the desirability of organizing such a section was called on December 31, 1896, and was signed by twenty-eight members of the Society, and the conference was attended by seventeen members. At this conference fourteen papers were read and at the second conference in April, 1897, ten papers were read. At the latter time the Chicago Section was formally established. S.

We are grieved again to record the death of another of our loyal subscribers and contributors, Joseph Carter Corbin, whose death occurred January 9, 1911, from heart failure. The following brief sketch of his life was furnished by his daughter Louise: Mr. Corbin was born in Chillicothe, Ohio, March 26, 1833. He was the eldest son of William and Susan C. Corbin; graduated from Ohio University in 1853, (A. M. from the same); married September 11, 1866, to Mary J. Ward, at Cincinnati; taught school in Louisville, Kentucky, about twelve years, assisting his brother-in-law, Rev. Henry Adams; later was employed by the Bank of Ohio Valley in Cincinnati; while there he served two years on the Board of Education, and edited *The Citizen* seven years. Moved to Little Rock, Arkansas, in 1870, and edited *The Republican* for General Powell Clayton; State Superintendent of Public Instruction in 1873; Principal of school at Jefferson City, Missouri, 1874-76; organized Branch Normal College at Pine Bluff, Arkansas, 1876, and was President 1876-1902; President Ouachita Baptist College, Camden, Arkansas, 1902 to January, 1904; at the request of the Board of Education of Pine

Bluff, he resigned and became Principal of Merrill School and served until the close of the school year, May, 1910. He was actively engaged in Masonic work and in conducting a private correspondence school at the time of his sudden death. He was the father of five children, two of whom, William H. and Louise M., survive him. In politics a Republican; in religion a Baptist; and a thirty-third degree Mason. He was a subscriber and contributor to the MONTHLY from its beginning.

BOOKS.

Diophantus of Alexandria. A Study in the History of Greek Algebra. By Sir Thomas L. Heath, K. C. B., Sc. D., sometime Fellow of Trinity College, Cambridge, England. Second edition, with a supplement containing an account of Fermat's Theorems and Problems connected with Diophantine Analysis and some Solutions of Diophantine Problems by Euler. Royal 8vo. Cloth, viii+388 pages. Price, 12s. 6d., net. London, England: The Cambridg University Press.

The first edition, the first English translation of Diophantus' *Arithmetica* which appeared in 1885, has long been out of print and thus that interesting and, to many, fascinating treaties had become practically inaccesible to English readers. Inquiries for it at different times suggested to the author the revision of his work done twenty-five years ago and the result is this second edition which conforms to recent discoveries in the history of mathematics. In this new edition, the author has rewritten the whole of the introduction, except the chapters on the editions of Diophantus, his methods of solutions, and the porisms, and other assumptions found in his work; while the chapters on the methods and on the porisms, etc., have been made fuller.

Much attention has been given to Fermat's notes on the various problems, and as much of this interesting material could not be incorporated as foot notes, the author added a supplement in which much attention is given to the theorems of Fermat and in which also is included a number of Euler's remarkable solutions of difficult Diophantine problems.

In its present form, the book will be in great demand by all mathematicians who are interested in this interesting phase of the theory of numbers. F.

College Mathematics Notebook. By Robert E. Moritz, Ph. D., Professor of Mathematics, University of Washington. Price, 80 cents. Ginn & Co.

This note book will be found very convenient and serviceable for the college teacher of mathematics. F.

ERRATA.

On page 241, No. 12, Vol. XVII, December 1910, problem 343 solution, make the following changes:

In line six of the solution for "A on the day x," read "A on the day $x+1$," and in the next line for "B on the day x," read "B on the day $x+1$." On page 243, in line three, for "$(1.05)^1$," read "$(1.05)^0$," and in the next line for "or," read "of," and in the same line for "0.202," read "0.002." In the fourth line from the end of solution for "(0.8583)," read "(0.9583)."

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TESTS OF SYMMETRIC POLYNOMIALS.

By DR. G. A. MILLER, University of Illinois.

In Bôcher's *Introduction to Higher Algebra*, 1907, page 240, the following theorem is stated: *A necessary and sufficient condition for a polynomial to be symmetric is that it be unchanged by every interchange of two variables.* This test becomes quite laborious even when the number of variables is reasonably small; for instance, for six variables there are fifteen distinct pairs of variables so that this theorem requires fifteen trials to prove that a polynomial in six variables is symmetric. In general, the number of trials according to the given theorem is clearly $\frac{n(n-1)}{2}$, and hence this number increases very rapidly with n. In view of this fact, it may be of interest to consider some tests requiring a very much smaller number of trials, especially since the determination of these tests is equivalent to a determination of sets of generating substitutions of the symmetric group of degree n and hence it is a problem which is common to at least two subjects.

If the following well known theorem: *A necessary and sufficient condition for a polynomial to be symmetric is that it be unchanged by every interchange of two variables such that one variable is the same in each of these pairs of variables,* were substituted for the one given above, the number of trials necessary to prove the symmetry of a polynomial of n variables would evidently be reduced to $n-1$, and the individual trials would not involve any more labor than those of the theorem of the first paragraph. Moreover, the proof of the theorem of the present paragraph involves very little more thought than that of the preceeding paragraph. In fact the $\frac{n(n-1)}{2}$ possible transpositions, or interchanges of pairs of variables formed from n variables, can clearly be obtained by transforming the $n-1$, which have a common variable by means of others of this set of $n-1$ transpositions. That every possible transposition on these n variables may be obtained in this manner, results from the following identity:

$$a_\alpha\, a_\beta = a_1\, a_\alpha\, .\, a_1\, a_\beta\, .\, a_1\, a_\alpha$$

If the polynomial is transformed into itself by a set of substitutions it must be invariant under the group generated by this set. As a symmetric polynomial is transformed into itself by all the possible substitutions on its variables, it results that we must always employ a set of generating substitutions of the symmetric group in proving that a polynomial is symmetric, as was remarked above. It is well known that every symmetric group is generated by two of its substitutions, and hence we do not need to employ more than two substitutions to prove or to disprove the symmetry of any polynomial. Moreover, since every symmetric group whose degree exceeds two is non-cyclic, we must always use at least two substitutions, when the number of variables exceeds 2, to prove or to disprove this symmetry.

Two generating substitutions of the symmetric group of degree n can be selected in a number of ways which increases rapidly with n. One such pair consists of an arbitrary cyclic substitution on the n variables and an arbitrary transposition not contained in a cycle of less than n variables in a power of this substitution. This fact may be stated more definitely as follows: *A necessary and sufficient condition that a cyclic substitution of degree n and a transposition on two of these n letters generate the symmetric group of degree n is that this transposition is not contained in any of the cycles of order less than n generated by this substitution.* The truth of this theorem is almost self-evident. It may, however, be regarded as a result of combining the following two well known theorems: If a primitive substitution group involves a transposition it is symmetric; a regular group having exactly k different sub-groups, besides the identity and the entire group, has also exactly k different systems of imprimitivity.*

One of the most useful theorems with respect to two generating substitutions of the symmetric group of degree n may be stated as follows: *If two cycles having only one common letter involve n different letters and if at least one of these cycles involves an even number of letters they must generate the symmetric group of the degree n.* The proof of this theorem results almost immediately from the fact that the commutator of the two given cycles is a cyclic substitution of order 3 according to a theorem due to Bochert.† The conjugates under these cycles of this substitution of order 3 generate the alternating group of degree n, and this alternating group together with the cycle of even order generates the symmetric group of degree n. The special case when the cycle which involves an even number of letters is a transposition, leads to the following corollary: *A necessary and sufficient condition for a polynomial in n variables to be symmetric is that it be unchanged by the cyclic interchange of some $n-1$ of these variables as well as by the interchange of the remaining variable and some one of these $n-1$ variables.*

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* Dyck, *Mathematische Annalen*, Vol. 22 (1883), p. 89.

† *Mathematische Annalen*, Vol. 33 (1889), p. 587.

It is a very simple matter to state numerous other criterions for the symmetry of polynomials of n variables. In Netto's *Theory of Substitutions*, 1892, page 90, it is stated that the probability that any two substitutions on n letters generate the symmetric group may be taken as about $\frac{3}{4}$. That is, if we interchange the n variables of a polynomial according to two substitutions, and if the polynomial remains unchanged, the probability that it is symmetric is about $\frac{3}{4}$. Although this implies that numerous tests for symmetry may readily be obtained, it is doubtful whether it is possible to find more useful general tests than those given in the preceding paragraphs. Among the most convenient additional tests is the following: *A necessary and sufficient condition for a polynomial in n variables to be symmetric is that it be unchanged by each of the $n-1$ interchanges which result when we interchange the first and the second variable, then the second and third, then the third and fourth, and finally the $(n-1)$th and the nth.*

Although the last theorem requires $n-1$ tests to prove the symmetry of a polynomial, each of these tests is so very simple that the total number of them involve about the same amount of labor as each of the sets of two tests given in the preceding theorems. In this connection it may be observed that the symmetric group of degree n can always be generated by a pair of substitutions of orders 2 and 3 respectively, except when n is one of the three numbers 5, 6, 8.* Hence we may prove the symmetry of a polynomial of any degree, besides these three, by means of two substitutions of orders 2 and 3 respectively. In practice this kind of general test is, however, less simple than some of those given above. On the other hand, the noted Italian mathematician, Alfredo Capelli, gave a test which is fairly convenient and can be easily stated as follows: *A necessary and sufficient condition for a polynomial to be symmetric is that it be unchanged by a cyclic interchange of all its variables as well as by the cyclic interchange of all but one of its variables, the variables having the same relative positions in the two cycles.*†

Deep interest in the symmetric functions was first aroused by the fact that the coefficients of the general equation of degree n, are symmetric functions of the roots. In 1770, Lagrange and others began the study of rational integral functions of n variables which are not necessarily symmetric, and observed that the number of different formal values assumed by such functions, when the variables are permuted in every possible way, is always a division of $n!$ Just as the permutations which do not affect the formal value of a symmetric polynomial, correspond to a substitution group, so there is also connected with each non-symmetric polynomial a substitution group which sheds light on its properties. In particular the number of different formal values of the polynomial, when its variables are permuted in every possible manner, is equal to the index of this substitution group, and the possibility of constructing polynomials with a given number of

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* *Bulletin of the American Mathematical Society*, Vol. 7 (1901), p. 426.

† Capelli, *Giornale di Matematiche*, Vol. 35 (1897), p. 354.

formal values, is completely determined by the possible substitution groups on these variables. The above remarks apply directly to all rational functions of the n variables as well as to the more special functions which are commonly (but not universally) called polynomials.

ON THE COMBINATION OF INVOLUTIONS.*

By D. N. LEHMER, University of California.

1. Given two involutions of rays in the same plane with centers at A and B a quadratic reciprocal transformation may be set up in the plane as follows: To any point P make correspond the intersection P' of the two rays at A and B which correspond in the involutions to the rays PA and PB. The point-to-point transformation thus defined is clearly involutorial. Certain points appear as exceptional in that the correspondence is not unique, namely: all of the points on the line joining A and B go by the transformation into the same point C. Further, all of the points on AC go into the same point B, and all of the points on BC go into the point A, as is seen by making the construction according to the definition. It will be suspected that the point C is the center of an involution which might be used instead of A or B to define the same transformation. That this is the case appears as a corollary from the following fundamental theorem.

2. Theorem. *If the point P describes a straight line the corresponding point P' describes a conic through A, B, and C.*

The point row P projects to A and B in two perspective pencils. The corresponding rays in the involutions at A and B are therefore projective and generate a conic through A and B. Since the point row P meets the line AB in one point the corresponding point P' goes through C. From the construction it appears further that the point row of the first order P is projective to the point row of the second order P'. It follows therefore that the pencils PC and $P'C$ are projective and that they are in involution. The point C is thus in all respects coördinate with A and B.

If we assume the theorem that a conic meets a curve of degree m in $2m$ points we may prove easily the more general theorem.

3. Theorem. *A curve of degree n goes by the above transformation into a curve of degree $2n$.*

For let the curve C of degree n go into a curve C'. Cut across C' by a line g. Transform C' and g. C' goes back into C and g goes into a conic γ. The points common to C and the conic γ, $2n$ in number, correspond to the points common to C' and the arbitrary line g.

* Presented at the September meeting of the San Francisco Section of the American Mathematical Society, under slightly different title.

4. In the above theorems we have assumed tacitly that the curve described by P does not pass through A, B, or C. If for instance in the first theorem the line traversed by P passes through A, the point P' must traverse the line corresponding to it in the involution at A. The conic in fact degenerates into this line and the line BC which corresponds to the point A. The more general theorem of paragraph 3 should be stated as follows to take account of this degeneracy. A curve of degree n which passes k times through the points A, B, or C goes into a degenerate curve of degree $2n$, which contains the sides of the triangle counted k times and a curve of degree $2n-k$. Thus a conic goes generally into a quartic which has A, B, and C for double points since the conic cuts the sides of the triangle generally in two points. If the conic goes through the point A the corresponding curve is a cubic with double point at A. If the conic contains two points A and B the corresponding curve is a conic through A, B, and C. If the conic passes through all three points the corresponding curve is a straight line.

5. The theory of the quadratic transformation is of course perfectly well known and many ways have been devised for realizing it geometrically.* The above method appears in many ways the simplest and most effective. Thus the four invariant points appear as the intersections of the focal rays at A with the focal rays at B; and since the focal rays at C must go through them they are presented as the vertices of a quadrilateral of which ABC is the diagonal triangle. Again, since AB and AC are corresponding rays in the involution at A that involution is determined by one pair of corresponding points P, P'. Two transformations S and T having the same base triangle ABC produce a collineation in the plane. For S throws a line into a conic through ABC, and T throws this conic into a line again. Conversely, it is easy to show that any collineation is the product in this way of two quadratic transformations. The vertices A, B, C, are the self-corresponding points in the collineation.

6. We have seen that two involutions A and B determine a third C. We will consider now three involutions A, B, and C which are not so related. It is clear that a point P in the plane will not generally have a point P' conjugate to it with respect to all three involutions. For the rays at A, B, and C which correspond to PA, PB, and PC will not generally meet in a point. Points may be found, however, which do have conjugates with respect to all three as the following theorem shows.

7. Theorem. *On any line in the plane there is at least one point and at most three which have conjugate points with respect to three unrelated involutions.*

For let the point P traverse the line g. The rays at A and B which correspond in the involutions at those points to PA and PB generate as we have seen a conic through A and B. There is also a conic through A and C. These two conics have a point A in common, and so have at least one and at

* See Reye, *Geometry of Position*, Holgate's Translation, p. 111.

most three other common points. These are easily seen to be points of the kind in question. Thus we have the following

Theorem. *The locus of points which have conjugate points with respect to three unrelated involutions in the plane is a cubic curve.*

8. If a point P is on the curve the point P' is also, as appears from the character of the correspondence. The points of the curve are thus paired in involution in the sense that four points P that project to A, B, or C in four harmonic rays correspond to four points P' that also project to A, B, or C in four harmonic rays. The points A, B, and C are easily seen to be points on the curve. To show this, construct the point of intersection C' of the rays at A and B which correspond to CA and CB. C and C' are conjugate points.

9. *If PP' and QQ' are two pairs of conjugate points so also are the points $(PQ, P'Q')=R$ and $(PQ', P'Q)=R'$.* For joining PP' and QQ' to A we get two pairs of corresponding rays in the involution at A. But if we project to A the opposite vertices of the complete quadrilateral PP', QQ', RR', we get three pairs of rays in the same involution. Thus the vertices RR' project to A and similarly to B and C in the involutions at those points.

10. *If P and P' are fixed and Q is taken arbitrarily on the curve, then QP and QP' meet the curve again in conjugate points R and R'.*

If Q moves so as to make R approach P then the conjugate R' must approach P' and we have at once the beautiful

Theorem. *The tangents to the curve at a pair of conjugate points meet again on the curve.*

In this case Q' lies on the line joining PP'.

11. Consider now four unrelated involutions of rays in the same plane. Is it possible to find points in the plane conjugate with respect to all four? Let the centers be A, B, C, and D. Points conjugate with respect to A, B, and C lie on a cubic through those points. Points conjugate with respect to A, B, and D lie on a cubic through those points. The two cubics have A and B for common points and also the point M, where M is the center of the third involution equivalent to, and determined by, the involutions at A and B, as in the first paragraph. Other intersections must occur in pairs, for if P lies on both cubics so also must P', which is conjugate to it with respect to all four involutions. If they have two pairs of points in common they must have a third pair. If they have still another pair it is possible to find as many pairs as one desires by the above quadrilateral construction. In that case the curves coincide and D lies on the cubic determined by A, B, and C. It follows that *the cubic may be generated by any three points on it as centers of involution,* provided, however, the three points are not centers of related involutions.

It follows from this that *any point on the cubic is joined to three pairs of conjugate points by three pairs of rays in involution.*

12. This last theorem is indeed the fundamental starting point for the

usual treatment of the general cubic by synthetic methods.*

13. We have omitted to speak of the corresponding discussion that might be made of point rows in involution. The extension of the method to space of three dimensions is of interest, however. Consider three quadric surfaces, A, B, and C. Then to any point in space P, we make correspond the point P' of intersection of the polar planes of P with respect to the three surfaces.† The correspondence is reciprocal and generally unique. By it *a straight line goes into a twisted cubic.* The polar planes of P describe projective axial pencils. If we cut across by an arbitrary plane we get three projective pencils of rays in that plane. By a discussion precisely similar to the discussion in paragragh 7 it is shown that one set of corresponding rays at least and three sets at most will meet in a point. These are points on the locus of P'.

14. By the above transformation *a plane goes into a cubic surface.* For suppose the plane μ goes into the surface M. Cut across M by an arbitrary line g. Reciprocate and the surface M goes into the plane μ and g goes into a twisted cubic which meets μ in one or three points. These are conjugate to the three points in which g meets M.

15. By an entirely similar argument, assuming the theorem: *A twisted cubic meets a surface of degree n in 3n points,* we may show that, *A surface of degree n goes into a surface of degree 3n.* Also by assuming that, *A cubic surface intersects a space curve of degree n in 3n points,* we may show that, *The above transformation throws a curve of degree n into a curve of degree 3n.*

16. We have seen in the corresponding plane theory that a straight line is transformed into a conic, and a conic into a quartic. The particular conic that corresponds to a straight line however transforms back not into a quartic but into the line. This was explained by the fact that the conic obtained from a straight line passed through each of the vertices of the triangle ABC and thus degenerated in the transformation into a quartic made up of four lines. A similar difficulty appears in the space theory. A plane γ goes into a cubic C, and this cubic C instead of going into a surface of degree nine, as the theory requires, goes back into the plane γ again. Also the cubic curve that corresponds to a straight line goes back into a straight line again and not into a curve of degree nine. To explain this difficulty, consider the plane γ and the corresponding cubic surface C_3. Cut across C_3 by any plane α. Reciprocate C_3 and α. C_3 goes into a plane γ, and α into a cubic surface A. Therefore the curve of intersection of α with C_3 goes into a cubic curve C lying in γ. Cut C_3 by a second plane β, and let the intersection line of α and β meet C_3 in three points P, Q, and R. As before, the curve of intersection of β and C_3 goes into a cubic curve b in γ. The two curves b and c have the three points P', Q', and R' in common. They have then generally six others, S_1, S_2, S_3, S_4, S_5, S_6. It is seen that S_1 corresponds to a point on

* See Schraeter, *Ebene Kurven Dritter Ordnung.* Leipzig, 1888.

† This correspondence is due to Steiner, *Gess. Werke, II,* p. 651.

α since it is a point on a, and also to a point on β since it is a point on b. But this means that the polar planes of S_1 meet in two distinct points and therefore all three polar planes meet not in a single point but in a line. There are six such points generally in any plane. *The locus of points therefore, such that their polar planes with respect to three quadric surfaces meet in a line, is a twisted sextic curve.* It is of course the Jacobian of the three surfaces.*

17. We note in passing that the six lines corresponding to the six points S in γ lie entirely in the surface C_3. Also that just as in the plane theory if a line goes through one of these points the corresponding twisted cubic degenerates into a conic and a straight line. Also if a line goes through two of these points the corresponding twisted cubic is made up of three straight lines. To the fifteen lines therefore joining the six points S in pairs, correspond fifteen other lines lying in the surface. Further, the conics through any five correspond to degenerate curves of degree six made up of six straight lines, five of which are the lines corresponding to the points S. The sixth also, therefore, lies on the surface giving five more or twenty-seven in all. The usual theorems may of course be made to follow.†

18. To a point on the sextic curve corresponds thus a line, and to any point on that line corresponds back again the point on the sextic. If then, a point P move along one of these lines the corresponding three axial pencils described by its polar planes have their axes meeting in the point on the sextic. The argument of paragraph 13 can be extended to show that if the axes of three projective pencils meet in a point there will be three sets of corresponding planes at most and one set at least that meet in the same straight line. This means that there are three positions for P on the line in question in which its three polar planes meet in a line. In other words, *Every line corresponding to a point on the sextic curve meets the sextic in one or three points.* From the proof we have established also that, *Through every point on the sextic curve will pass one or three lines of the sort in question.* It is of interest then to investigate the singly infinite system of lines which correspond to points on the Jacobian sextic of three quadrics.

19. To obtain the degree of the surface O of which these lines are the generators we observe first that a straight line must meet it in eight points. For the twisted cubic corresponding to the line must cut the Jacobian sextic eight times. Otherwise the twisted cubic would go back into a curve of degree higher than one. The curve of degree nine into which this twisted cubic must go is thus a degenerate curve made up of the original line and eight lines of the surface O. Again a plane goes into a cubic surface. This cnbic surface contains every point of the sextic since the plane from which it is derived cuts every line of the surface O. The cubic surface goes back into a plane instead of into a surface of degree nine. This means that the surface O is of the eighth degree.

* See Salmon's *Geometry of Three Dimensions*, p. 213.

† Reye, l. c., Chapter 26.

20. This surface O is thus a ruled surface of degree eight, every ruling of which meets the Jacobian in three points (or one) and such that three (or one) rulings go through every point on the sextic. The Jacobian is thus a triple line on the surface, three sheets of the surface passing through it. A plane section of the surface will thus give a curve of degree eight with six triple points.

21. This surface O and the Jacobian sextic of the three quadric surfaces will serve to explain the discrepancies in the degrees of reciprocal curves and surfaces. Thus a curve of degree n goes by the transfarmation into a curve of degree $3n$. But since the curve of degree n meets the surface O in $8n$ points, the curve of degree $3n$ meets the sextic in $8n$ points. When therefore, we reciprocate *this* curve of degree $3n$ part of the resulting curve is made up of $8n$ lines of the surface O, and the degree of the resultant curve is thus $9n-8n$ or n, which is the original curve. Similarly for surfaces. Every ruling of O meets a surface of degree n in n points, so that the reciprocal surface of degree $3n$ passes n times through the Jacobian curve. In reciprocating the surface back again we get a surface of degree $9n$ to be sure, but the surface O is n times a part of it. The reciprocal surface is thus of degree $9n-8n$ or n, the original surface.

PROOF OF THE FIRST FORMULA FOR EVALUATING 0/0.

By N. J. LENNES, Columbia University, New York.

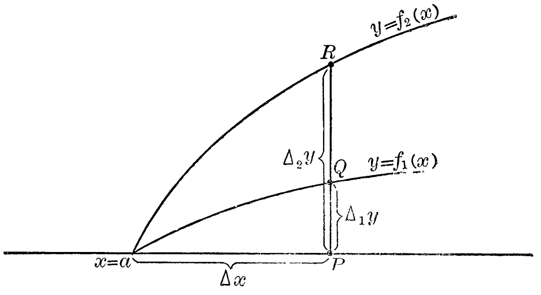
It is well known that many proofs of theorems, while entirely logical and conclusive, afford little or no insight into the theorems proved and certainly furnish no obvious clue to the origin either of the theorem or the proof. On the other hand it is by no means infrequent that one has direct insight into the nature of a theorem or mathematical process and that such insight may be of great value both as an aid to the memory and in judging the true value and uses of the theorem or process while at the same time it may be found impossible, or at least impracticable, to formulate the elements of thought which constitute this insight into a logical proof.

It is believed to be a pedagogical principle of wide application that, other things being equal, the one of two modes of treatment of any topic is to be preferred which affords the more direct and obvious connection between intuitional insight on the one hand and rigorous logical proof on the other. For this reason the proofs given below seem to be worth publishing.

Theorem. *If $f_1(x)$ and $f_2(x)$ are continuous functions each possessing a derivative at $x=a$, and such that $f_1(a)=0$, $f_2(a)=0$, while $f_2'(a)\neq 0$, then*

$$\lim_{x \doteq a} \frac{f_1(x)}{f_2(x)} = \frac{f_1'(a)}{f_2'(a)}.$$

Proof. Using the notation indicated in the figure,



$$f_1'(a)=\underset{\Delta x \doteq 0}{\text{Lim.}}\frac{\Delta_1 y}{\Delta x}$$

and $f_2'(a)=\underset{\Delta x \doteq 0}{\text{Lim.}}\frac{\Delta_2 y}{\Delta x}$,

and hence $\frac{f_1'(a)}{f_2'(a)}$

$$=\underset{\Delta x \doteq 0}{\text{Lim.}}\frac{\frac{\Delta_1 y}{\Delta x}}{\frac{\Delta_2 y}{\Delta x}}. \quad \text{But } \frac{\frac{\Delta_1 y}{\Delta x}}{\frac{\Delta_2 y}{\Delta x}}=\frac{\Delta_1 y}{\Delta_2 y}=\frac{f_1(x)}{f_2(x)}.$$

$$\text{Hence, } \frac{f_1'(a)}{f_2'(a)}=\underset{x \doteq a}{\text{Lim.}}\frac{f_1(x)}{f_2(x)}.$$

In case $f_2'(a)=0$ and also $f_1'(a)=0$. the more general theorem is established in the usual manner as follows:*

By the generalized mean value theorem; namely, that if $f_1(x)$ and $f_2(x)$ have derivatives on the interval ab, then there is a value of x, X, on ab, such that

$$\frac{f_1(a)-f_1(b)}{f_2(a)-f_2(b)}=\frac{f_1'(X)}{f_2'(X)}.$$

Since in the case of our theorem we have $f_1(a)=0$ and $f_2(a)=0$ it follows that in the above figure

$$\frac{f_1(b)}{f_2(b)}=\frac{f_1'(X)}{f_2'(X)},$$

or, what is the same thing,

$$\frac{f_1(x)}{f_2(x)}=\frac{f_1'(X)}{f_2'(X)},$$

* See, for instance, Osgood's *Calculus*, pp. 234, 235.

where X is some point between a and x. Hence, obviously,

$$\underset{x \doteq a}{\text{Lim.}} \frac{f_1(x)}{f_2(x)} = \underset{x \doteq a}{\text{Lim.}} \frac{f_1'(X)}{f_2'(X)}.$$

If we now assume $f_2''(a) \neq 0$, we have, by the theorem first proved,

$$\underset{x \doteq a}{\text{Lim.}} \frac{f_1(x)}{f_2(x)} = \underset{x \doteq a}{\text{Lim.}} \frac{f_1'(x)}{f_2'(x)} = \frac{f_1''(a)}{f_2''(a)}.$$

Obviously, by the same process, in case $f_2''(a)=0$, but $f_2'''(a) \neq 0$,

$$\underset{x \doteq a}{\text{Lim.}} \frac{f_1(x)}{f_2(x)} = \frac{f_1'''(a)}{f_2'''(a)},$$

and so on for the higher cases.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

348. Proposed by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

Prove the following relation between Jacobi's symbols:

$$\left(\frac{d}{n}\right) = (-1)^{m(d-1)/2}\left(\frac{d}{2md-n}\right)$$

where d and n are positive odd numbers and $2md > n$.

I. Solution by the PROPOSER.

Replace each symbol by the value given by the generalized reciprocity theorem; then on the right apply

$$\left(\frac{2md-n}{d}\right) = \left(\frac{-n}{d}\right) = (-1)^{(d-1)/2}\left(\frac{n}{d}\right).$$

Hence the relation will be true if

$$\frac{n-1}{2}\cdot\frac{d-1}{2}\equiv\frac{m(d-1)}{2}+\frac{d-1}{2}\cdot\frac{2md-n-1}{2}+\frac{d-1}{2}\pmod 2,$$

or,

$$\frac{(d-1)}{2}[m+md-n+1]\equiv 0\pmod 2.$$

But the first factor is an integer, and the second is an even integer.

II. Solution by BENJAMIN FRANKLIN YANNEY, A. M., Mount Union College, Alliance, Ohio.

An implied assumption is that d and n are relatively prime to each other. It follows, then, that d and $2md-n$ are also relatively prime to each other.

1. By the reciprocity theorem, $\left(\frac{d}{n}\right)\left(\frac{n}{d}\right)=(-1)^{\frac{1}{2}(d-1).\frac{1}{2}(n-1)}$.

2. Multiply by $\left(\frac{-1}{d}\right)=(-1)^{\frac{1}{2}(d-1)}$, member by member:

$$\left(\frac{d}{n}\right)\left(\frac{n}{d}\right)\left(\frac{-1}{d}\right)=(-1)^{\frac{1}{2}(d-1).\frac{1}{2}(n-1)+\frac{1}{2}(d-1)}=(-1)^{\frac{1}{2}(d-1).\frac{1}{2}(n+1)}.$$

But, $\left(\frac{n}{d}\right)\left(\frac{-1}{d}\right)=\left(\frac{-n}{d}\right)=\left(\frac{2md-n}{d}\right)$, as is well known. [See (35) and (36), p. 60, Bachmann's *Neuere Zahlentheorie.*]

3. Therefore, $\left(\frac{d}{n}\right)\left(\frac{2md-n}{d}\right)=(-1)^{\frac{1}{2}(d-1).\frac{1}{2}(n+1)}$.

4. Multiply by $\left(\frac{d}{2md-n}\right)$:

$$\left(\frac{d}{n}\right)\left(\frac{2md-n}{d}\right)\left(\frac{d}{2md-n}\right)=(-1)^{\frac{1}{2}(d-1).\frac{1}{2}(n+1)}\left(\frac{d}{2md-n}\right);$$

$$\text{or, }\left(\frac{d}{n}\right)(-1)^{\frac{1}{2}(2md-n-1).\frac{1}{2}(d-1)}=(-1)^{\frac{1}{2}(d-1).\frac{1}{2}(n+1)}\left(\frac{d}{2md-n}\right).$$

5. Multiply by $(-1)^{\frac{1}{2}(2md-n-1).\frac{1}{2}(d-1)}$ and reduce:

$$\left(\frac{d}{n}\right)=(-1)^{md(d-1)/2}\left(\frac{d}{2md-n}\right).$$

6. $\therefore\left(\frac{d}{n}\right)=(-1)^{m(d-1)/2}\left(\frac{d}{2md-n}\right)$, since $(-1)^{md(d-1)/2}=(-1)^{m(d-1)/2}$, d being an odd integer.

349. Proposed by JOSEPH A. NYBERG, Student, University of Chicago.

To show that the determinant of the nth order:

$$D_n=\begin{vmatrix} C & -1 & 0 & 0 & 0 & 0 & 0 & . & . \\ -1 & C & -1 & 0 & 0 & 0 & 0 & . & . \\ 0 & -1 & C & -1 & 0 & 0 & 0 & . & . \\ 0 & 0 & -1 & C & -1 & 0 & 0 & . & . \\ 0 & 0 & 0 & -1 & C & -1 & 0 & . & . \\ 0 & 0 & 0 & 0 & -1 & C & -1 & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{vmatrix}$$

has the value: $D_n=C^n+\sum_{r=1}^{n}(-1)^r\frac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!}C^{n-2r}$.

I. Solution by the PROPOSER.

If the determinant is expanded by the elements of the first row or column, we get:

$$D_n=C.D_{n-1}-D_{n-2},$$

where D_{n-1} and D_{n-2} are determinants similar to D_n but having, respectively, one and two fewer rows and columns. Compare

(1) $$D_n+D_{n-2}=C.D_{n-1}$$
with
(2) $$\sin(a+b)+\sin(a-b)=2\sin a\cos b.$$

Put $a=n\theta$, $b=\theta$. Then (2) becomes

$$\sin(n+1)\theta+\sin(n-1)\theta=2\sin n\theta\cos\theta.$$

Now put
(3) $$C=2\cos\theta \text{ and } D_n=c\sin(n+1)\theta,$$

where c is independent of n. To determine its value we note that, for $n=1$,

$$D_1=C=2\cos\theta\equiv\frac{\sin 2\theta}{\sin\theta}.$$

(4) $\therefore c=\csc\theta$.

As a result of (3) and (4) we can write:

$$D_n=\frac{\sin(n+1)\theta}{\sin\theta}.$$

But, from Chrystal's *Algebra*, Vol. 2, p. 252, we have:

$$(5)\quad \frac{\sin(n+1)\theta}{\sin\theta}=(2\cos\theta)^n+\sum_{r=1}^{n}(-1)^r\frac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!}(2\cos\theta)^{n-2r}$$

Using equations (5) and (3) the desired formula is obtained.

A second solution, though quite long, and therefore will only be sketched here, may be given as follows: We have

$$(1)\qquad \begin{aligned} D_i&=C^i+a_{1i}\,C^{i-1}+a_{2i}\,C^{i-2}+a_{3i}\,C^{i-3}+\ldots\\ D_{i+1}&=C^{i+1}+a_{1,\,i+1}\,C^i+a_{2,\,i+1}\,C^{i-1}+a_{3,\,i+1}\,C^{i-2}+\ldots\\ D_{i+2}&=C^{i+2}+a_{1,\,i+2}\,C^{i+1}+a_{2,\,i+2}\,C^i+a_{3,\,i+2}\,C^{i-1}+\ldots \end{aligned}$$

But since $D_{i+2}=C.D_{i+1}-D_i$, we have

$$(2)\qquad D_{i+2}=C^{i+2}+a_{1,\,i+1}\,C^{i-1}+(a_{2,\,i+1}-1)\,C^i+(a_{3,\,i+1}-a_{1,\,i})\,C^{i-1}+\ldots$$

Comparing (2) with the last of (3), it can be shown that $a_{jp}=0$ when j is odd. Consequently the powers of C diminish by 2.

By studying the successive coëfficients of powers of C we can show by induction that the coefficient of $C^{(n-2r)}$ is

$$(-1)^r\frac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!},$$

which then gives the formula as originally stated.

II. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology.

Expanding in terms of the first column, we have the following relation connecting three determinants of the kind here considered whose orders are n, $n-1$, $n-2$:

$$D_n=CD_{n-1}-D_{n-2}.$$

Forming some of the successive values of D we find:

$$\begin{aligned} D_1&=C,\\ D_2&=C^2-1,\\ D_3&=C^3-2C,\\ D_4&=C^4-3C^2+1, \end{aligned}$$

$$D_5=C^5-4C^3+3C,$$
$$D_6=C^6-5C^4+6C^2-1,$$
$$D_7=C^7-6C^5+10C^3-4C,$$
$$D_8=C^8-7C^6+15C^4-10C^2+1.$$

It is clear that starting with D_n and reading the terms diagonally we have the expansion of $(C-1)^n$. For instance, starting with D_4 we have $C^4-4C^3+6C^2-4C+1$. Hence, reading horizontally, the 1, 2, 3, 4, etc., terms of D_n will be the 1, 2, 3, 4, etc., terms in the expansions of $(C-1)^n$, $(C-1)^{n-1}$, $(C-1)^{n-2}$, $(C-1)^{n-3}$, etc., respectively. The rth term will be the rth term of the expansion of $(C-1)^{n-r+1}$. Hence

$$D_n=C^n-(n-1)C^{n-2}+\frac{(n-2)(n-3)}{2!}C^{n-4}-\frac{(n-3)(n-4)(n-5)}{3!}C^{a-6}+\ldots$$

$$\text{or } D_n=C^n+\sum_{r=1}^{n}(-1)^r\frac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!}C^{n-2r}.$$

A very neat solution of this problem was also received from a contributor who failed to sign his name to the solution. Will contributors please note that we wish them to put their names to the solutions and to observe the order of the printed solutions, viz., put name at beginning of solution rather than at the end?

GEOMETRY.

369. Proposed by W. J. GREENSTREET, A. M., Editor, Mathematical Gazette, Stroud, England.

Prove by inversion that if two circles cut at a given angle, touch each a given circle, and pass each through the same fixed point, then shall the envelope of the points of contact be a conic.

Discussion by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Since all of the contact points lie on the fixed circles it seems probable that the desired locus is that of the second point of intersection of the variable circles. In Fig. 1, let S_1 and S_2 be the fixed circles, T_1 and T_2 the variable circles passing through the fixed point Q at the constant angle θ.

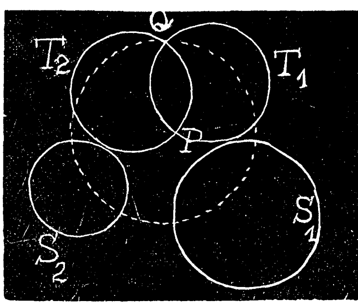


Fig. 1.

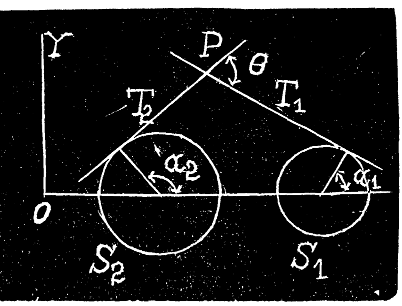


Fig. 2.

Then the locus to be found is that of the second intersection P. Through the fixed circles and Q considered as a point circle, Fig. 1, there may be passed one orthogonal circle whose center is their radical center. With Q as the center of inversion and with any convenient radius, Fig. 1 is to be inverted, thus changing T_1 and T_2 into straight lines in Fig. 2, crossing at the constant angle θ, and tangent to the new fixed circles S_1 and S_2.

The orthogonal circle mentioned above inverts into the straight line through the centers of S_1 and S_2.

The analytic work from this stage is as follows: The equations of S_1 and S_2 may be taken as

$$(x-a_1)^2+y^2=r_1^2 \text{ and } (x-a_2)^2+y^2=r_2^2,$$

hence the equations of T_1 and T_2 are

$$(x-a_1)\cos\alpha_1+y\sin\alpha_1=r_1 \text{ and } (x-a_2)\cos\alpha_2+y\sin\alpha_2=r_2,$$

in which $\alpha_1-\alpha_2=\theta=$constant, this being the angle condition. Replacing θ by 2ϕ and writing $\alpha_1=\alpha+\phi$, $\alpha_2=\alpha-\phi$, the equations of T_1 and T_2 become

$$(x-a_1)\cos(\alpha+\phi)+y\sin(\alpha+\phi)=r_1, \quad (x-a_2)\cos(\alpha-\phi)+y\sin(\alpha-\phi)=r_2.$$

Expanding the trigonometric functions and solving the two equations for $\cos\alpha$ and $\sin\alpha$ leads to the elimination of α, and gives the following for the locus of P:

$$\begin{aligned}&\{[(x-a_1)(x-a_2)+y^2]\sin\theta-y(a_1-a_2)\cos\theta\}^2\\ &=r_1^2[(x-a_2)^2+y^2]+r_2^2[(x-a_1)^2+y^2]\\ &-2r_1r_2\{[(x-a_1)(x-a_2)+y^2]\cos\theta+y(a_1-a_2)\sin\theta\}.\end{aligned}$$

Two special cases of the preceding equation may best be considered at this time. If S_1 and S_2 in Fig. 2 are coincident or even merely concentric, then the locus of P is a circle by elementary geometry, hence the locus of P in Fig. 1 is also a circle. This case occurs when Q, the fixed point in Fig. 1, is either of the point circles of the family of circles having a common radical axis determined by S_1 and S_2. From the general equation this result appears by placing $a_1=a_2$, and then the locus degenerates into a point circle and the circle mentioned above. Again the general equation degenerates when $r_1=r_2=0$, giving

$$\{[(x-a_1)(x-a_2)+y^2]\sin\theta-y(a_1-a_2)\cos\theta\}^2=0,$$

which is the circle, doubly counted, resulting from making S_1 and S_2 poin circles in both figures. As in the preceding case, the locus of P in Fig. 1 is also a circle. The general locus is a bicircular quartic, hence its inverse is also a bicircular quartic, thus determining the locus of P (Fig. 1) which constitutes the solution of the given problem.

NOTE. No solutions of 374 and 376 have yet been received. We shall be pleased to have our contributors take up these two problems for solution. ED. F.

CALCULUS.

300. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve the differential equation obtaining the complete primitive,

$$(x^2+x^2y+2xy-y^2-y^3)dx+(y^2+xy^2+2xy-x^2-x^3)dy=0.$$

Solution by V. M. SPUNAR, Chicago, Ill., and the PROPOSER.

If F is an integrating factor of the differential equation

$$Mdx+Ndy=0,$$

we have the relation

$$\frac{F'(v)}{F(v)}=\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N\dfrac{\partial v}{\partial x}-M\dfrac{\partial v}{\partial y}}=\frac{4(x-y)(x+y+1)}{N\dfrac{\partial v}{\partial x}-M\dfrac{\partial v}{\partial y}}.$$

On account of the symmetry of M and N, it is evident that v is symmetrical in x and y. Trying $v=x+y$, we have

$$\frac{F'(v)}{F(v)}=-\frac{4(x+y+1)}{(x+y)^2+2(x+y)}=-\frac{4(v+1)}{v^2+2v}.$$

Integrating, we have,

$$F=\frac{1}{v^2(v+2)^2}=\frac{1}{(x+y)^2(x+y+2)^2}.$$

Trying, also, $v=(1+x)(1+y)$, we have,

$$\frac{F'(v)}{F(v)}=-\frac{2}{x+xy+y}=-\frac{2}{v-1}.$$

Integrating,

$$F=\frac{1}{(v-1)^2}=\frac{1}{(x+xy+y)^2}.$$

If we have two integrating factors, their ratio, put equal to a constant, is the complete primitive.

Therefore, the complete primitive is

$$\frac{(x+y)^2(x+y+2)^2}{(x+xy+y)^2}=c,$$

or, extracting the square root,

$$\frac{(x+y)\,(x+y+2)}{x+xy+y}=c_1,$$

or, in the form,

$$\frac{x^2+y^2}{x+xy+y}=c_2.$$

301. Proposed by C. N. SCHMALL, New York City.

Show that the volume of the surface, $\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}+\left(\frac{z}{c}\right)^{\frac{2}{5}}=1$, is $\frac{100\,\pi\,abc}{3.5.7.11.13}$.

I. Solution by FRANCIS RUST, C. E., Pittsburg, Pa.

Volume is special case of Derichlet's special formula:

$$\int\int\int\ldots dx\,dy\,dz\ldots x^{l-1}.y^{m-1}.z^{n-1}\ldots$$

$$=\frac{a^l.b^m.c^n\ldots}{p.q.r\ldots}\cdot\frac{\Gamma\left(\frac{l}{p}\right).\,\Gamma\left(\frac{m}{q}\right).\,\Gamma\left(\frac{n}{r}\right)\ldots.}{\Gamma\left(1+\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+\ldots.\right)},$$

provided, $\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r+\ldots=1$, and the integration to extend over all *positive* values of the variables.

In this case,

$$V=\int\int\int dx\,dy\,dz=\frac{abc}{p.q.r}\cdot\frac{\Gamma\left[\frac{1}{p}\right].\Gamma\left[\frac{1}{q}\right].\Gamma\left[\frac{1}{r}\right]}{\Gamma\left[1+\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right]}$$

with $p=q=r=\frac{2}{5}$. The formula yields $\frac{1}{8}$ part of the total volume, which consequently is

$$V=8.\tfrac{125}{8}.abc.\frac{[\Gamma(\frac{5}{2})]^3}{\Gamma(\frac{17}{2})}=\frac{20.\pi}{3.7.11.13}.abc.$$

II. **Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, and V. M. SPUNAR, Chicago, Ill.**

The solid is evidently symmetrical with respect to the axes.

Solving, we find $z=c\left[1-\left(\dfrac{x}{a}\right)^{\frac{2}{5}}-\left(\dfrac{y}{b}\right)^{\frac{2}{5}}\right]^{\frac{5}{2}}.$

When $z=0$, $y=b\left[1-\left(\dfrac{x}{a}\right)^{\frac{2}{5}}\right]^{\frac{5}{2}}=w$, for brevity; then

$$z=c\left[\left(\frac{w}{b}\right)^{\frac{2}{5}}-\left(\frac{y}{b}\right)^{\frac{2}{5}}\right]^{\frac{5}{2}}=\frac{c}{b}[w^{\frac{2}{5}}-y^{\frac{2}{5}}]^{\frac{5}{2}}.$$

$$V=8\int_0^a dx\int_0^w\frac{c}{b}[w^{\frac{2}{5}}-y^{\frac{2}{5}}]^{\frac{5}{2}}\,dy.$$

Let $y=w\sin^5\theta$, $dy=5w\sin^4\theta\cos\theta\,d\theta$.

$$V=\frac{40c}{b}\int_0^a dx\int_0^{\frac{1}{2}\pi}w^2\cos^6\theta\sin^4\theta\,d\theta=\frac{40c}{b}\int_0^a w^2\,dx\left[\frac{\sin^5\theta\cos^5\theta}{10}\right.$$

$$\left.+\tfrac{1}{256}(3\theta-\sin4\theta+\frac{\sin8\theta}{8})\right]_0^{\frac{1}{2}\pi}=\tfrac{60}{256}\frac{c}{b}\pi\int_0^a b^2\left[1-\left(\frac{x}{a}\right)^{\frac{2}{5}}\right]^5dx$$

$$=\tfrac{60}{256}.\frac{256}{7.9.11.13}\pi\,abc=\frac{20\,\pi\,abc}{3.7.11.13}.$$

Also solved by J. Scheffer and the Proposer.

MECHANICS.

251. Proposed by J. G. ROSE, B. A. (Oxion), Mt. Angel College, Oregon.

ABC is a uniform triangular lamina of weight $3W$ such that $AB=2AC$. A particle of weight W is attached to it at C. Show that if the lamina be suspended from angle A, it will rest with AB and AC equally inclined to the vertical.

Solution by J. EDWARD SANDERS, U. S. Weather Bureau, Columbus, Ohio.

Draw the median AOD, AO being two thirds of AD, the vertical AP, P and D on BC, and OE and CF, perpendicular to AP. Let $AC=b$, $AB=2b$, $\angle CAB=\alpha$, $\angle CAD=\theta$, and $\angle CAP=\phi$.

Now $BC^2=AC^2+AB^2-2AC.AB\cos\alpha=5b^2-4b^2\cos\theta$.
$\therefore CD=\frac{1}{2}b\sqrt{(5-4\cos\alpha)}$, and $4AD^2=2(AC^2+AB^2)-BC^2$
$=5b^2+4b^2\cos\alpha$.
$\therefore AD=\frac{1}{2}b\sqrt{(5+4\cos\alpha)}$, and $AO=\frac{1}{3}b\sqrt{(5+4\cos\alpha)}$.

Also $\cos\theta=\dfrac{AC^2+AD^2-CD^2}{2AC.AD}$

$$=\frac{b^2+\frac{1}{4}b^2(5+4\cos\alpha)-\frac{1}{4}b^2(5-4\cos\alpha)}{2b.\frac{1}{2}b\sqrt{(5+4\cos\alpha)}}=\frac{1+2\cos\alpha}{\sqrt{(5+4\cos\alpha)}}.$$

$$\therefore \sin\theta=\frac{2\sin\alpha}{\sqrt{(5+4\cos\alpha)}}.$$

Since the uniform weight, $3W$, acts at the centroid O, we have, by taking moments about A,

$$W.b\sin\phi=3W.\tfrac{1}{3}b\sqrt{(5+4\cos\alpha)}\sin(\theta-\phi).$$

$$\begin{aligned}\therefore \sin\phi&=\sqrt{(5+4\cos\alpha)}\sin(\theta-\phi).\\ &=\sqrt{(5+4\cos\alpha)}(\sin\theta\cos\phi-\cos\theta\sin\phi)\\ &=2\sin\alpha\cos\phi-(1+2\cos\alpha)\sin\phi\\ &=2\sin\alpha\cos\phi-2\cos\alpha\sin\phi-\sin\phi\\ &=\sin(\alpha-\phi).\end{aligned}$$
$\therefore \phi=\frac{1}{2}\alpha$.

Also solved by J. Scheffer and S. G. Barton.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

Edited by Dr. G. E. Wahlin, University of Illinois.

174. Proposed by B. KRAMER, Student, University of Pittsburg, Pittsburg, Pa.

Find a general solution of $x(x+a)=y^2$, a, x, and y being integers. Given a, required to find x to satisfy conditions.

II. Solution by E. B. ESCOTT, Ann Arbor, Mich.

The solution by Prof. F. L. Griffin is defective. His statsment that $\dfrac{n^2}{a-2n}$ is integral only when n is a multiple of $a-2n$ is not true.

For example, if $a=15$, $n=3$, $\dfrac{n^2}{a-2n}=1$, but $\dfrac{n}{a-2n}=\frac{1}{3}$.

Correct solution. Let the highest common divisor of x and $x+a$ be m. Then evidently the solution is

$$x=mr^2\ldots(1),$$
$$x+a=ms^2\ldots(2),$$
$$y=mrs\ldots(3).$$

From (1) and (2), $a=m(s^2-r^2)$.

Then if a is given, we can put s^2-r^2 equal to the different factors of a, and m equal to the remaining factor. s^2-r^2 may be put equal to any factor of a excepting the double of an odd number.

Example. $a=15$. If $a^2-r^2=15$, $s+r=15$ or 5, $s-r=1$ or 3,

$$\left.\begin{matrix} s=8 \\ r=7 \end{matrix}\right\} \text{ or } \left.\begin{matrix} s=4 \\ r=1 \end{matrix}\right\},\ m=1.$$

These give $x=49$, $y=56$, $x=1$, $y=4$.

This last solution is omitted in Professor Griffin's table owing to his incorrect analysis.

If $s^2-r^2=5$, $m=3$, $x=12$, $y=18$.

If $s^2-r^2=3$, $m=5$, $x=5$, $y=10$.

179. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve the equation in integers, $x^n+y^n+z^n+xyz=100x+10y+z$.

Solution by A. H. HOLMES, Brunswick, Maine.

Put $xyz=100x$. $\therefore yz=100$, and $z=100/y$.

$\therefore x^n+y^n+\dfrac{100^n}{y^n}=10y+\dfrac{100}{y}$; or $x^n=10y+\dfrac{100}{y}-y^n-\dfrac{100^n}{y^n}$.

For n even, $10y+\dfrac{100}{y}> y^n+\dfrac{100^n}{y^n}$.

Put $n=2$. Then $10y+\dfrac{100}{y}> y^2+\dfrac{10000}{y^2}$.

But $y=\dfrac{100}{z}=1, 2, 4, 5, 10, 20, 25, 50,$ or 100. $\therefore n=1$ and $x=9y$.

$$\therefore n=1:\ \left|\begin{array}{l} x=9,\ 18,\ 36,\ 45,\ 90,\ 180,\ 225,\ 450,\ 900. \\ y=1,\ 2,\ 4,\ 5,\ 10,\ 20,\ 25,\ 50,\ 100. \\ z=100,\ 50,\ 25,\ 20,\ 10,\ 5,\ 4,\ 2,\ 1. \end{array}\right.$$

Put $xyz=10y$. $\therefore xz=10$, $z=\dfrac{10}{x}$ and $x^n+y^n+\dfrac{10^n}{x^n}=100x+\dfrac{10}{x}$ or $y^n=100x+\dfrac{10}{x}-x^n-\dfrac{10^n}{x^n}$.

Put $n=1$. Then $y=99x$. But $x=10/z=1, 2, 5, 10$.

$$\therefore n=1: \left| \begin{array}{l} x=1,\ 2,\ 5,\ 10, \\ y=99,\ 198,\ 495,\ 990. \\ z=10,\ 5,\ 2,\ 1. \end{array} \right.$$

Put $n=2$. Then $y^2=100x+10/x-x^2-100/x^2$. $\therefore x=1$ and 10.

$$\therefore n=2: \left| \begin{array}{l} x=1,\ 10. \\ y=3,\ 30. \\ z=10,\ 1. \end{array} \right.$$

In neither case, when $y=1, 2, 4, 5, 10, 20, 25, 50, 100$, and $x=1, 2, 5, 10$, can $n=3$.

180. Proposed by A. H. HOLMES, Brunswick, Maine.

Find integral values for x and y in the following: $96x-96y+21=\square$.

Solution by the PROPOSER.

If we solve the equation

$$(1) \quad (24y+z)^2=25(24x+z-1),$$

for z we find

$$z=\frac{25-48y\pm5\sqrt{(96x-96y+21)}}{2}.$$

Since the coefficient of z^2 in (1) is unity, if for a pair of values of x and y, $96x-96y+21$ is a square, the corresponding value of z must be a rational integer. But no set of integral rational values of x, y, z will satisfy (1), for if the value of z is odd, the left hand member of (1) is odd and the right hand member is even, and vice versa for z even. Hence there are no integral values for x and y such that the given expression is a square.

☞Problems and solutions for this department should be sent to Dr. Wahlin, Urbana, Ill.

PROBLEMS FOR SOLUTION.

ALGEBRA.

353. Proposed by DANIEL KRETH, Oxford, Iowa.

Divide 2940 into two such factors that the square of one factor minus 21 will equal three times the other factor.

354. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

To find x in the following equation:

$$0.002\{6x-20[(1.05)^x-1]\}=0.012\{21[(1.05)^{x-1}-1]-(x-1)\}.$$

GEOMETRY.

384. Proposed by S. LEFSEHETZ, Clark University.

Let ABC be a triangle, O a circle tangent to its three sides, T a variable tangent of O, which cuts the sides BC, CA, AB in a, b, c. Oa', Ob', Oc' the perpendiculars in O to Oa, Ob, Oc, cutting, respectively, T in points a', b', c'. Prove that Aa', Bb', Cc' meet in a point t, and find the locus of t when T varies. Purely geometrical proofs wanted.

385. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Given a triangle ABC, find the radius of a circle touching two of its sides and a line parallel to the third, at a distance $d=u+2r$.

386. Proposed by DANIEL KRETH, Oxford, Iowa.

Construct the triangle, having given, the vertical angle, the sum of the three sides, and the perpendicular.

CALCULUS.

308. Proposed by C. N. SCHMALL, New York City.

Prove, by calculus, that of all isoperimetric triangles, the equilateral has the greatest area.

309. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology.

In practical problems involving maxima and minima, it is really the greatest and least values of the function which are desired. Show why we can assume that the maximum is the greatest value and the minimum the least value under the conditions.

310. Proposed by C. N. SCHMALL, New York City.

Evaluate $\int_0^\pi \frac{dx}{1-2a\cos x+a^2}$. Edwards' *Integral Calculus for Beginners*, page 131, ex. 9, (iii). The answer given is $\frac{\pi}{1-a^2}$. Is this a complete answer to the question?

MECHANICS.

260. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

To the ends of a fine inextensible string, length $2l$, are attached to equal, smooth, spherical, equally elastic (e) particles. At first the middle point of the string touches a rigid, fixed, circular rim, radius a, and the particles are $2l$ apart. They are now projected with equal velocities perpendicular to the string and curl around the rim. If l is greater than πa, find the condition that the particles will move after collision along tangents to the rim, the whole motion being on a smooth horizontal plane.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

183. Proposed by M. T. GOODRICH, Dixfield, Maine.

Show what relation must exist between the quantities A, B, and C, in the harmonic ratio $\frac{AB}{(A+B+C)(-C)}=-1$, so that they will be real positive integers.

NOTES AND NEWS.

Professor E. B. Van Vleck is again in residence at the University of Wisconsin, not at the University of Minnesota, as inadvertantly announced last month. S.

Editor Slaught left the University of Chicago March 21st, to make a trip through the South, in the interest of the University. He will make a number of speeches before college students and educational organizations and will reach home about the first week in April. F.

Dr. A. M. Hiltebeitel has been appointed instructor in mathematics at the University of Pennsylvania. Dr. H. B. Smith has been appointed instructor in the same department for the ensuing term, to fill the vacancy caused by the temporary absence of Professor Evans. S.

The Annals of Mathematics will be taken over by Princeton University after the completion of the present volume. This journal was founded at the University of Virginia, under whose auspices it was published until it was taken over by Harvard University a few years ago. S.

A splendid edition of Halsted's Rational Geometry, in French, is issuing from the presses of Gauthier-Villars, the famous publishing house of Paris. In the history of geometry this is a fine compliment to Dr. Halsted, as we believe no geometry of the Western Hemisphere has ever before been so honored. F.

The syllabus on calculus, forming the fourth section of the report of the committee on mathematics for students of engineering, of which Professor Huntington, of Harvard University, is chairman, will appear in the forthcoming number of the *Bulletin* of the Society for the Promotion of Engineering Education. The committee presented a preliminary report at the joint meeting of engineers and mathematicians at Minneapolis last December, and it is hoped to present a completed report at the coming summer meeting of the Society for the Promotion of Engineering Education. S.

Three important reports of American Committees under the International Commission on the teaching of mathematics, have been printed in the *Bulletin of the American Mathematical Society*. namely: (1) On the preparation of instruction for colleges and universities, in Volume XVII, No. 2; (2) On university courses in mathematics and the Master's degree, in Volume XVII, No. 5; (3) Preparation for research and the Doctor's degree in mathematics, in Volume XVII, No. 6. These reports and all the others of the American Committee will be published later by the Department of Education at Washington. S.

The April meeting of the American Mathematical Society will be held at the University of Chicago, on Friday and Saturday, April 28-29. At this meeting Professor Maxime Bôcher will deliver his Presidential address, the provisional title of which is: "Charles Sturm's Published and Unpublished Work on Differential and Algebraic Equations." The announcement states that: "Except for the summer meetings, this is the first gathering of the whole society that has been held since 1896, and it is hoped to make it a memorable occasion. The full attendance of the Chicago Section should be reinforced by large delegations from both East and West." S.

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RITHMOMACHIA, THE GREAT MEDIEVAL NUMBER GAME.

By DAVID EUGENE SMITH and CLARA C. EATON.

When the subject of number games shall be adequately treated, and the long and interesting story is told of how the world has learned to handle the smaller numbers quite as much through play as through commerce, the climax will probably be found in the chapter relating to the Battle of Numbers, the Rithmomachia of the Middle Ages. For here was a tournament worthy of intellectual foes, a play that outranked chess as much as chess surpasses mere dicing, and a game that was by its very nature closed to all save selected minds that had been trained in the Boethian arithmetic, the Latinized Nicomachus, the last great effort in the Pythagorean philosophy of numbers.

But when this story comes to be told the one who relates it will have no easy task, and the object of this paper is rather to set forth the problem than to solve it. For although we have manuscripts of three writers of the eleventh century, two of the twelfth, one of the thirteenth, and Bradwardin's work of the fourteenth,* and although we have several printed treatises on the subject,† we know practically nothing of the origin of the game. We only know that the medieval writers attributed it to Pythagoras, that no trace of it has been discovered in Greek literature, and that no mention of it has been found before the time of Hermannus Contractus (1013-1054). The name, which appears in a variety of forms,‡ points to a Greek origin, the more so because Greek was little known at the time when the game first appears in literature. Based as it is upon the Greek theory of numbers,

* There is a twelfth century manuscript of Hermannus Contractus (1013-1054) at Paris, and others of later date in various libraries. Wappler has published this, and also a treatise by Asilo (before 1077), with part of an anonymous one of the twelfth century. Odo also wrote on the subject in the eleventh century. Peiper has edited Fortolfus's work of the twelfth century. Several other manuscripts are known. Consult Wappler, in the *Zeitschrift für Mathematik und Physik*, Vol. 37, p. 1 (1892), and Peiper in the *Abhandlungen*, Vol. 3 (1880).

† We have made free use of the brief description given by Jacobus Faber Stapulensis (1496), possibly from Shirewood's manuscript, and the works of Boissière (French edition 1554, Latin edition 1556) and Barozzi (1572). Abraham Riese (1562) published Asilo's manuscript.

‡ Correctly, Rithmomachia, but also in such incorrect forms as Rythmomachia (Battle of Rhythms), Rithmimachia, Rythmimachia, etc.

appearing as it does with a Greek name, necessarily a game known to but few and one that would naturally pass from the élite to the élite, attracting no attention from the populace, it is easy to feel that the origin of the game is to be sought in the Greek civilization, and perhaps in the later schools of Byzantium or Alexandria. The very fact that a game so well known as to justify printed treatises in Latin, French, Italian, and German, in the sixteenth century, and to have public advertisements of the sale of the board and pieces under the shadow of the old Sorbonne,* and that this game has been forgotten for upwards of three centuries of modern civilization, shows how easily it might have left no record during the period which we so truly designate as the dark ages. Furthermore the early manuscripts are so obscure and condensed as to show that they presupposed a knowledge of the game, merely summarizing some of the more difficult rules to be followed, so that it would seem a proper conjecture that scholars were transmitting it by word of mouth, only recording now and then a few directions that were not so easily retained in the memory.

The game was played on a board resembling the one used for chess or checkers, with eight squares on the shorter side, but with sixteen on the longer side. The forms used for the pieces were triangles, squares, circles, and pyramids, and the pieces were set as is shown in the illustration (Fig. 1) here given from Boissière. In this setting the white pieces are numbered in the same way as the black pieces in the work by Jacobus Faber Stapulensis, so that the color had no significance. The names of the pieces and their position before the opening of the game is shown in Fig. 2, the lower ones being called the Evens (starting from the even numbers 2, 4, 6, 8), and the upper ones being called the Odds. The dots placed below the numbers serve to mark the bottom of the piece, so that 6 shall not be confused with 9, 81 with 18, etc.

The first row on each side (Fig. 2) is made of the odd and even numbers, respectively, unity being admitted as not a number *"sed fons et origo numerorum."* The second row is made up of the squares of the first. The sum of the two rows gives the first row of triangles. The second row of triangles is formed from the first one by means of a relation known as superparticularis; that is, a number in the second row is found by joining the corresponding number in the first row to an aliquot part of it determined by the number in the first circle at the top. For example, 81 is obtained from 72 by adding $\frac{1}{8}$ of 72, 8 being the number in the circle at the top of that column. Similarly, $42=36+\frac{1}{6}$ of 36, $20=16+\frac{1}{4}$ of 16, and $6=4+\frac{1}{2}$ of 4. The ratio of the aliquot parts added, to the numbers at the top, are therefore $\frac{3}{2}$ (sesquialtera), $\frac{5}{4}$ (sesquiquarta), etc. The relation of the lower triangles of each side to the lower circles of the other side does not seem to have been noticed. The first row of squares is formed by adding the respective triangles ($9+6=15$, $25+20=45$, etc.), but one square on

* One of the editions of Boissière advertises the sale of this material at the shop of "John the Gentile."

each side (91 for the Evens, and 190 for the Odds) is replaced by a pyramid. These pyramids are formed by superposing squares (thin prisms), and are used to call attention to the peculiar construction of the numbers which they represent. Thus $91=6^2+5^2+4^2+3^2+2^2+1$, and $190=8^2+7^2+6^2+5^2+4^2$. Since the former contained the squares of all numbers from 1 to 6, it was called a perfect pyramid, but since the latter lacked $3^2+2^2+1^2$ it was known as *tricurta* (thrice curtailed),

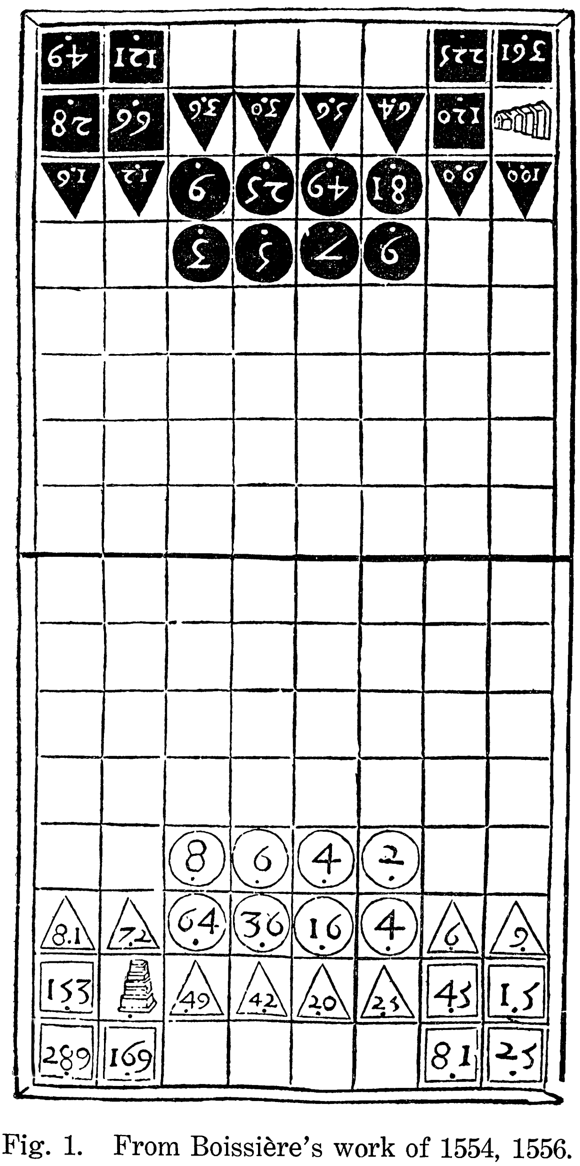


Fig. 1. From Boissière's work of 1554, 1556.

The lower row of squares is obtained from the upper one by a formula somewhat like the one used in obtaining the lower row of triangles. In the case of the squares, if we call the number in the circle at the top n, and the number in the upper square s, then the number in the lower square is $\left(\frac{2n+1}{n+1}\right)s$. Thus $\left(\frac{2\cdot 2+1}{2+1}\right)\cdot 15=\frac{5}{3}\cdot 15=25$; $\left(\frac{2\cdot 4+1}{4+1}\right)\cdot 45=\frac{9}{5}\cdot 45=81$, and so on. The ratio between 25 and 15, $\frac{5}{3}$, is one of the superpartientes, namely, the superbipartientes (surpassing by two parts). It should be said, however, that other rules are given for the derivation of these numbers, and that there are slight variations in the pieces, but these have no significance.

The pieces are now arranged for the opening of the play in the manner shown in Fig. 1. The players move the pieces in turn, as in chess. A circle moves one space, a triangle three, and a square four. The game consists in capturing an opponent's pieces, this being effected in one of four ways—by meeting, by assault, by ambuscade, and by siege.

The method by meeting may be illustrated as follows: If Even's triangle 25 can, by advancing three spaces, reach Odd's circle 25, Even does not move his piece, but simply takes up his opponent's.

The capture by assault is affected in this way: If a smaller number, multiplied by the number of vacant spaces between it and a larger one equals the larger one, it may take it. For example, Odd's circle 5 may take Even's square 45 if nine separates the two. This requires the players to be familiar with the multiplication table, and for this purpose Fortolfus provides the usual square array known in the Middle Ages as the *mensa Pythagorica.*

The capture by ambuscade is as follows: If two pieces whose sum equals the number on an opponent's piece can be moved into the spaces on either side of it, the latter is ambuscaded and must surrender. For example, to capture Odd's triangle 12, Even's circles 4 and 8 must be able to move on either side of it.

The capture by siege is effected if a piece is immediately surrounded on all four sides by opposing pieces; that is, if the adjacent spaces above, below, to the right, and to the left are filled.

It is evident that a pyramid can rarely be taken except by siege. Interest was therefore added to the game by making it subject to several attacks. A pyramid was considered to be in danger whenever one of its laminæ was attacked by any one of the four methods. In this case a ransom was allowable, namely, a piece of the same value as the base. In case no such piece could be offered because of prior capture, any other piece could be given that the opponent might be willing to accept. Positive capture of the piece not being possible if the numbers 91 and 190 were retained, it was permitted if the base square was successfully attacked, namely, 36 or 64. The piece having no particular value, its loss was no more serious than that of any other piece, but the plans of attack were more interesting.

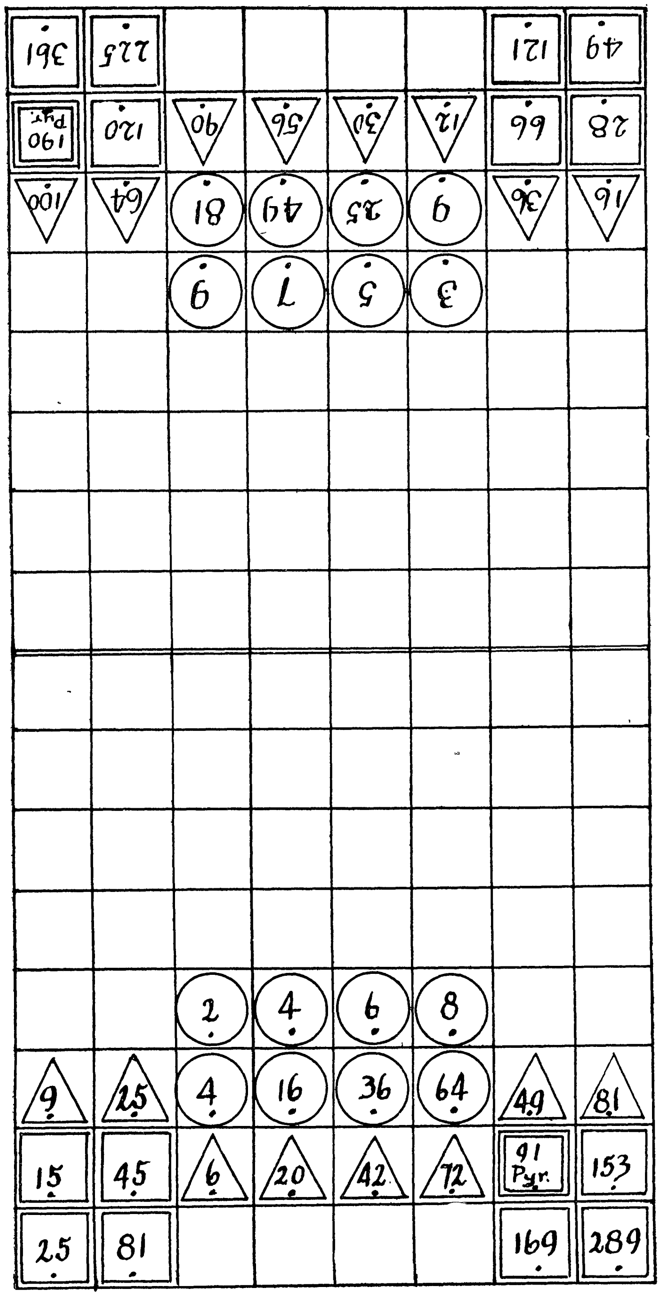


Fig. 2. The Setting of the Pieces.

As already stated, the game consists in capturing an opponent's pieces. This, however, is not all there is of it. The capture is undertaken for the purpose of obtaining what is technically called a Victory, and the rules provide for no less than eight of these Victories. Before beginning to play, the particular kind of Victory for which the contest is to be waged is agreed upon by the parties. Five of these kinds are known as Common Victories, and the rest as Proper Victories, the former being considered as suited to tyros and the latter as worthy of veteran players.

Common Victories were, as already said, of five kinds, as follows: (1) Victory *de corpore*, decided by the number of pieces captured; (2) Victory *de bonis*, depending upon the value of the pieces; (3) Victory *de lite*, depending not only upon the value of the pieces but upon the number of the digits inscribed upon them; (4) Victory *de honore*, depending upon both the number of the pieces and their value; (5) Victory *de honore liteque*, depending upon the number of pieces, their value, and the number of digits inscribed upon them.

If, for example, the players decided upon the Victory *de corpore*, they would agree in advance upon some number, as twenty-four, as the winning number. As soon as either player captured twenty-four of the opponent's pieces he won the game.

If they decided in advance upon the Victory *de bonis*, they would agree upon some number like 160 as the winning number. Each player would then seek to capture pieces of which the sum of the values should equal or exceed 160.

If they decided upon the Victory *de lite* they might again select 160 with the further condition that the total number of digits on the pieces should equal some small number such as eight. A player would then try to capture pieces like 56, 64, 28, and 15, but would not try for 121, 9, and 30.

If the victory was to be *de honore* the players would agree upon some number like 160 for the sum of the values, and some other number like five for the number of the pieces. In this case neither 56, 64, 28, 12 nor 121, 9, 30 would suffice, but 64, 36, 30, 25, 5 would meet the two conditions.

In the Victory *de honore liteque* the players might agree upon 160 for the values, five for the number of pieces, and nine for the number of digits. These conditions are satisfied by 64, 36, 30, 25, 5 from the Odd's pieces, and 64, 36, 42, 16, 2 from the Even's pieces.

It is already evident that the game has more of merit than at first seemed probable. These Common Victories do not, however, show it as played by the real lover of Rithmomachia, It was in the Proper Victories that he found a game worthy of his efforts, and with these we shall close this description.

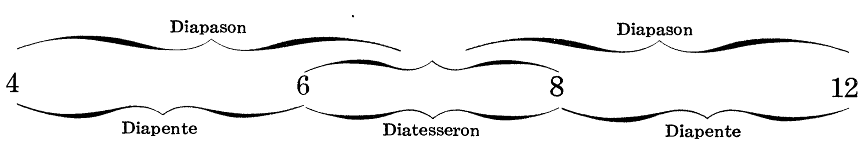
The Proper Victories were known by the names of Magna, Major, and Praestantissima, and they resulted from combinations relating to the three best known types of progressions, the arithmetic, geometric, and harmonic—

progressions that had come down through the Greek mathematics from the Pythagoreans. In each of these victories the pieces, one of which must be taken from the opposing side, must be displayed in the selected progression from the opponent's side of the board.

The Victoria Magna consists in arranging three counters in any one of the three simple progressions. There are forty-one combinations that make possible such an arrangement in arithmetic progression, eighteen in geometric progression, and seventeen in harmonic progression. The possibilities are greater for Even in the first case and for Odd in the second case, and they are equal in the third case. One of these arrangements, in geometric progression, is 6, 8, 12. To this Fortolfus gave the name Cubic Victory, the first number representing the faces, the second the vertices, and the third the edges of a cube.

The earlier writers gave to the second of the Proper Victories the name Victoria Minor, but Boissière calls it Victoria Major. In this there are combined two progressions, arithmetic and geometric, geometric and harmonic, or harmonic and arithmetic. To secure this victory four pieces must be brought in line in the enemy's field, two of which must belong to one of the selected progressions and two to the other. For example, 2, 3, 4, 8 would gain a Victoria Major for either Even or Odd, for 2, 3, 4 are in arithmetic progression and 2, 4, 8 are in geometric, where 2, 4, 8 are Even's pieces and 3 is Odd's piece. There are in all sixty-one such double progressions, all but one of which can be used by Even, and all of which can be used by Odd.

The climax of the game was reached in the Victoria Praestantissima, or Victoria Excellentissima. In this victory it was necessary to get four numbers in a row, which numbers embodied all three progressions. There are only six possible solutions to this problem, namely, (2, 3, 4, 6), (4, 6, 8, 12), (7, 8, 9, 12), (4, 6, 9, 12), (3, 5, 15, 25), (12, 15, 16, 20). Of these only one (4, 6, 9, 12) contains four terms in geometric progression, the others containing proportions. Upon these combinations the early writers dilate with not a little affection. For example, in the set 4, 6, 8, 12, the comparison of 12 and 8, or of 6 and 4, is a sesquialtera ($\frac{3}{2}$), corresponding to the fifth in music; 8 and 4, or 12 and 6, have the ratio 2:1, that of the octave or diapason. The ratio 8:6 gives the diatesseron, while 12:4 gives the interval, a twelfth, including both diapason and diapente. The ratio 8:(6—4) gives the interval of two octaves, or a fifteenth, and all of this was shown in graphic form as follows:



Such is a brief description of the game of which Boissière speaks as "*Noblissimus et antiquissimus ludus Pythagoreus qui Rythmomachia nominatur,*"* a game of which we are told many of the devotees were men of no mean reputation. Such leaders of thought as Gerbert, whom his contemporaries called a wizard but made a Pope; Hermannus, whose infirmity gave him the name of Contractus, by which he is commonly known; Robertus Castrensis, who helped to make the Arab learning known;† Nicolaus Horem, who became Bishop of Lisieux in 1377, and who sought to enrich the intellectual world by his teaching of the ancient theory of numbers; Oronce Finé (Orontius Finaeus), who was professor of mathematics in the (later called) Collége de France in 1532; Jacques le Fèvre d'Estaples (Jacobus Faber Stapulensis), the learned tutor of the son of François I; Thomas Bradwardin, who died in 1349 as Archbishop of Canterbury, and who, from his great learning, was known as "Doctor Profundus;" John Shirwood (Shirewode), who died in 1494 as Bishop of Durham—these are the names of some of those who played the game, and several of them composed tractatês setting forth its merits. It cannot be revived, since the interest in the number theory for which it stood has passed away, but even some slight understanding of its nature cannot fail to have interest for any one who takes pleasure in mathematics, in education, or in the evolution of both mathematics and education from the ideals of the Greek philosophy to the ideals of the present day.

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* For the full title in facsimile, see Smith, *Rara Arithmetica*, Boston, 1909, p, 272. with other references in the index.

† Professor Karpinski of the University of Michigan, is now working on one of his translations, the algebra of Al-Khowarazmi.

IDEALS OF A QUADRATIC NUMBER FIELD IN CANONIC FORM.

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Sommer* shows that any ideal of the quadratic field $k(\sqrt{m})$ may be reduced to a canonic form $(i,\ i_1+i_2 w)$, i, i_1, and i_2 being rational integers, i_2 a factor of both i and i_1, and

$$w\dagger=\begin{cases}\sqrt{m}, \text{ when } m \text{ is not congruent to } 1\ (4)\\ \dfrac{1+\sqrt{m}}{2}, \text{ when } m\equiv 1\ (4).\end{cases}$$

Any integer‡ of the ideal may be expressed in the form

$$li+n(i_1+i_2 w),$$

l and n being rational integers.

Using a slightly different notation, let us write the canonic form

$$r(s,\ t+w)$$

r, s, and t being rational integers, $r\neq 0$, and (to make the form unique)

$$s>t \geqq 0 \tag{1.}$$

$s=1$, $t=0$ would give the canonic form of the rational principal ideal (r), and $r=1$, $s=1$, $t=0$ would give the unit ideal containing every integer of the field.

Consider the case $r=1$, or the ideal $(s,\ t+w)$. s is the highest common factor of all the rational integers of the ideal,§ and it follows at once that t must satisfy the relation

$$\begin{cases} t^2\equiv m(s)\\ (2t+1)^2\equiv m(4s)\end{cases} \tag{2.}$$

Conversely, an ideal $(s,\ t+w)$ is in canonic form if s and t are rational in-

* *Vorlesungen über Zahlentheorie.* Leipzig, B. G. Teubner, 1907. See pp. 36-44. These are the ideals conceived by Dedikind and Treated by Dirichlet and others.

† Whenever double lines are written with a brace throughout this paper, the upper line will be for the case m not congruent to 1 (4) and the lower one for the case m congruent to 1 (4).

‡ The word "integer," unqualified, will be used in this paper to mean a quadratic integer.

§ Sommer, loc. cit., p. 40.

tegers satisfying conditions (1.) and (2.); for it is readily seen that, under these conditions, any integer of the field, $(a_1+b_1w)s+(a_2+b_2w)(t+w)$, may be expressed in the form $ls+n(t+w)$, l and n being rational integers.

The ideal conjugate to $(s, t+w)$, is, in canonic form,

$$\begin{cases} (s,\ s-t+w) \\ (s,\ s-t-1+w). \end{cases}$$

From conditions (1.) and (2.) and well-known theorems in the theory of rational integers, it follows that, for a given prime p not a factor of the discriminant of the field $\binom{d=4m}{d=m}$, there are two canonic ideals $(p, t+w)$, or none, according as

$$\begin{cases} \left(\dfrac{m}{p}\right)=1 \text{ or } -1 \\ \left(\dfrac{m}{4p}\right)=1 \text{ or } -1 \end{cases}$$

When there are two, they are conjugate. If p is a factor of d, there is always one and only one ideal $(p, t+w)$. It is self-conjugate; and $t=0$ except in the two cases,

$$\text{(1.) } m\equiv 1\ (4), \text{ when } t=\frac{p-1}{2},$$
$$\text{and (2.) } m\equiv 3\ (4) \text{ and } p=2, \text{ when } t=1.$$

For any given rational integer s, there will be ideals of the form $(s, t+w)$ if and only if m is a quadratic remainder for each prime factor of s. If there are any such ideals, there will be just 2^r where r is the number of *distinct* prime factors of s which are not factors of d.

MULTIPLICATION.

Consider the product $(p, t_1+w)(p, t_2+w)$, p being a rational prime not a factor of d. Either these ideals are conjugate, in which case their product is the principal ideal $p(1, w)$, or else $t_1=t_2$ and the product is the square $(p, t_1+w)^2$. If $(p, t_1+w)^2=(s, t+w)$, s must be a number of $(p, t+w)$* and must therefore be divisible by p. Also, since p^2 is a number of $(s, t+w)$, s must be a factor of p^2. Hence s is either p or p^2. If $s=p$, then $(p, t_1+w)^2$ is (p, t_1+w) or its conjugate $(p, p-t_1+w)$.† The

* If one ideal is a factor of another, then every integer of the second is an integer of the first: cf. Sommer, loc. cit., p. 46.

† The argument is given for the case m not congruent to 1 (4). A very similar argument disposes of the case m congruent to 1 (4).

first supposition is trivial, being possible only for $(p,\ t_1+w)=(1,\ w)$. If $(p,\ t_1+w)^2=(p,\ p-t_1+w)$, then $p-t_1+w$ would be an integer of the ideal $(p,\ t_1+w)$. This could only be true if p were a factor of $2t_1$; and since $p\neq 2$ and $p>t_1$, it is impossible. Hence $s=p^2$, and t is uniquely determined by the conditions

$$\begin{cases} t^2\equiv m(p^2) \\ (2t+1)^2\equiv m(4p^2) \end{cases}$$
$$t\equiv t_1(p)$$
$$\text{and}\quad 0 \leqq t<p^2.$$

The condition $t\equiv t_1(p)$ must be true because pt_1+pw must be an integer of the product ideal, and hence there must be rational integers l and n such that

$$pt_1+pw=lp^2+n(t+w)$$

and it is evident that $n=p$ and $l=\dfrac{t_1-t}{p}$.

Similarly, it may be shown that

$$(p,\ t_1+w)^n=(p^n,\ t+w),$$

t being uniquely determined by the conditions

$$\begin{cases} t^2\equiv m(p^n) \\ (2t+1)^2\equiv m(4p^n) \end{cases}$$
$$t\equiv t_1(p)$$
$$0 \leqq t<p^n.$$

If q is a prime factor of d, then $(q,\ t+w)$ is self-conjugate; $(q,\ t+w)^2=q(1,\ w)$, $(q,\ t+w)^{2n}=q^n(1,\ w)$, and $(q,\ t+w)^{2n+1}=q^n(q,\ t+w)$.

Consider next the product $(s_1,\ t_1+w)(s_2,\ t_2+w)$ where s_1 and s_2 are relative primes. If the product is $(s,\ t+w)$, s must be an integer of both the ideals $(s_1,\ t_1+w)$ and $(s_2,\ t_2+w)$, and hence must be divisible by both s_1 and s_2, and therefore by their product s_1s_2. But s_1s_2 is an integer of the ideal $(s,\ t+w)$, and hence is divisible by s. Therefore $s=s_1s_2$, and t is uniquely determined by the conditions

$$\begin{cases} t^2\equiv m(s_1s_2) \\ (t2+1)^2\equiv m(4s_1s_2) \end{cases}$$

$$t \equiv t_1 (s_1)$$
$$t \equiv t_2 (s_2)$$
$$0 \overline{\overline{<}} t < s_1 s_2.$$

Consider the ideal $(s,\ t+w)$, and let s_1 be any factor (prime or not) of s. Determine t_1 by the conditions $t_1 \equiv t(s_1)$ and $0 \overline{\overline{<}} t < s_1$. Then it may be readily shown that every integer of the ideal $(s,\ t+w)$ is also an integer of the ideal $(s_1,\ t_1+w)$, and hence $(s_1,\ t_1+w)$ is a factor of $(s,\ t+w)$. We can then find a prime ideal factor of $(s,\ t+w)$ for every prime factor of s; and if we multiply these prime factors together by the laws shown above, their product will evidently be $(s,\ t+w)$. It follows then that *the ideal $(s,\ t+w)$ has a prime ideal factor for every prime factor of s, and that it has no other factors.*

Consider now the general product of $r_1(s_1,\ t_1+w)$ and $r_2(s_2,\ t_2+w)$, and let it be written $r_1 r_2 \{r(s,\ t+w)\}$. In general, s_1 and s_2 will contain some prime factors which are factors of d, and some which are not; also some prime factors common to both s_1 and s_2, and some appearing only in one of the s's. Let the factors not in d be denoted by p's, and those in d by q's; and let the factors common to s_1 and s_2 be indicated by a bar thus, $\bar{p}$, $\bar{q}$. We may then write

$$s_1 = p_1^{\alpha_1}\ p_2^{\alpha_2}\ p_3^{\alpha_3} \dots q_1\ q_2 \dots \bar{p}_1^{\beta_1}\ \bar{p}_2^{\beta_2} \dots \bar{q}_1\ \bar{q}_2 \dots$$

$$\text{and } s_2 = p_{h+1}^{\alpha_{h+1}}\ p_{h+2}^{\alpha_{h+2}} \dots q_{k+1}\ q_{k+2} \dots \bar{p}_1^{\gamma_1}\ \bar{p}_2^{\gamma_2} \dots \bar{q}_1\ \bar{q}_2 \dots$$

(q's may not occur to powers higher than the first).

Every factor p^α or q not common to both s_1 and s_2 will be a factor of s; and will contribute, for the determination of t, the condition

$$t \equiv t_1 \text{ or } t_2 \ (p \text{ or } q) \tag{3.}$$

Every $\bar{q}$ will be a factor of r, and will not appear as a factor in s.

For each $\bar{p}$ we first determine whether $t_1 \equiv t_2\ (\bar{p})$ or t_1 not congruent to $t_2\ (\bar{p})$. Those $\bar{p}$'s which are in the second class we will denote by a double bar $\bar{\bar{p}}$. For $\bar{p}$'s for which the congruence holds, $\bar{p}^{\beta+\gamma}$ will be a factor of s, $\bar{p}$ will not appear in r, and we will have the condition

$$t \equiv t_1 \equiv t_2 \ (\bar{p}) \tag{4.}$$

For a $\bar{\bar{p}}$, let δ be the smaller of the numbers β and γ. Then $\bar{\bar{p}}^\delta$ will be a factor of r, $\bar{\bar{p}}^{|\beta-\gamma|}$ will be a factor of s, and we will have

$$
\left.\begin{array}{l} t \equiv t_1 \ (\bar{\bar{p}}), \text{ if } \beta > \gamma \\[2ex] \text{or } t \equiv t_2 \ (\bar{\bar{p}}), \text{ if } \gamma > \beta \end{array}\right. \tag{5.}
$$

The factors thus determined will make up r and s, and t will finally be uniquely determined by the condition

$$
\left\{\begin{array}{l} t^2 \equiv m \ (s) \\ (2t+1)^2 \equiv m \ (4s) \end{array}\right. \tag{6.}
$$

together with conditions (3.), (4.), and (5.) above. Hence the product of $r_1(s_1,\ t_1+w)$ and $r_2(s_2,\ t_2+w)$ may be written $r_1 r_2 r(s,\ t+w)$, in which

$$
r = \bar{q}_1 \ \bar{q}_2 \ \cdots \ \bar{\bar{p}}_1^{\,\delta_1} \ \bar{\bar{p}}_2^{\,\delta_2} \ \cdots
$$

$$
s = p_1^{\alpha_1} \ p_2^{\alpha_2} \ \cdots \ q_1 \ q_2 \ \cdots \ p_{h+1}^{\alpha h+1} \ p_{h+2}^{\alpha h+2} \ \cdots \ q_{k+1} \ q_{k+2} \ \cdots \ \bar{p}_1^{\,\beta_1+\gamma_1} \ \bar{p}_2^{\,\beta_2+\gamma_2} \ \cdots \ \bar{\bar{p}}_1^{\,|\beta_1-\gamma_1|} \ \bar{\bar{p}}_2^{\,|\beta_2-\lambda_2|} \ \cdots
$$

and t is given by the conditions (3.)...(6.)

Examples. Consider the product $(399,\ 182+\sqrt{7})(378,\ 175+\sqrt{7})$. Here $d=28$, $s_1=399=19.3.7$, $s_2=378=2.3^3.7$, $s=19.2.3^2$; $t\equiv 175(2)$, $t\equiv 175(3)$, $t\equiv 182(19)$, $t^2\equiv 7(19.2.3^2)$, and $0 \leqq t < (19.2.3^2)$, which makes $t=103$; $r=7.3$. Hence the product is $21(342,\ 103+\sqrt{7})$.

Consider $(120,\ 43+w)(700,\ 91+w)$ in the field $k(\sqrt{[-111]})$, $m\equiv 1(4)$, $w=\dfrac{1+\sqrt{[-111]}}{2}$, $d=-111$. Here $s=3.7.2^5.5$, $r=5$, and $t=1771$ from the conditions $t\equiv 43\equiv 91(2)$, $t\equiv 43(3)$, $t\equiv 91(5)$, $t\equiv 91(7)$, $(2t+1)^2\equiv -111(13440)$, giving the product $5(3380,\ 1771+w)$.

Canonic Form of the Principal Ideal Corresponding to an Irrational Integer.

The principal ideal corresponding to the integer $\alpha+\beta w$, α and β being rational integers prime to each other, may be put into canonic form by taking s as the norm of $\alpha+\beta w$, *i. e.*,

$$
s = \left\{\begin{array}{l} \alpha^2 - \beta^2 m \\ \alpha^2 + \alpha\beta - \beta^2 \dfrac{m-1}{4}. \end{array}\right.
$$

Then, for m not congruent to 1 (4), find a and b such that $a\beta+b\alpha=1$, and hence $(a+bw)(\alpha+\beta w)=a\alpha+b\beta+w$, and we may take $t\equiv a\alpha+b\beta m(s)$. For

the case $m \equiv 1$ (4), find a and b such that $(a+b)\beta+b\alpha=1$, and take $t \equiv a\alpha + b\beta \frac{m-1}{4} (s)$.

But, given an ideal in canonic form, it is not readily determined whether it is or is not a principal ideal; and hence we look for the necessary and sufficient conditions that a given ideal $(s,\ t+w)$ may be the principal ideal corresponding to some integer $\alpha+\beta w$. Consider the relation

(7.) $ls+nt+nw=(a+bw)(\alpha+\beta w)$, l, n, a, b, α, and β all rational integers.

Evidently $(s,\ t+w)$ will be a principal ideal if, and only if, we can find α and β such that

(1.) When a and b are arbitrarily assigned, l and n can be found to satisfy (7.); and

(2.) When l and n are arbitrarily assigned, a and b can similarly be found.

For simplicity consider the case m not congruent to 1 (4). Equating rational and irrational parts of (7.) we have

$$n=b\alpha+a\beta$$
$$l=\frac{a(\alpha+\beta t)+b(\beta m-\alpha t)}{s},$$

and if l is to be an integer for all values of a and b, we must have

$$\alpha-\beta t \equiv 0 \ (s) \qquad (8.)$$
$$\beta m-\alpha t \equiv 0 \ (s). \qquad (9.)$$

Again,
$$a=\frac{-l\alpha s-n(\beta m-\alpha t)}{\alpha^2-\beta^2 m}$$
$$\text{and } b=\frac{l\alpha s+n(\beta-\beta t)}{\alpha^2-\beta^2 m}$$

and if a and b are to be integers for all values of l and n, we must have

$$\alpha s \equiv 0 \ (\alpha^2-\beta^2 m) \qquad (10.)$$
$$\beta s \equiv 0 \ (\alpha^2-\beta^2 m) \qquad (11.)$$
$$\alpha-\beta t \equiv 0 \ (\alpha^2-\beta^2 m) \qquad (12.)$$
$$\beta m-\alpha t \equiv 0 \ (\alpha^2-\beta^2 m) \qquad (13.)$$

If (8.) is true, then $\alpha t-\beta t^2 \equiv 0$ (s), and since $t^2 \equiv m$ (s) we must have $\alpha t-$

$\beta m \equiv 0$ (s). Hence (8.) includes (9.). Again, if α contained any factor of $\alpha^2-\beta^2 m$, β would also have to contain it; and since α and β are prime to each other, this is impossible. Hence (10.) and (11.) reduce to $s\equiv 0$ $(\alpha^2-\beta^2 m)$, and this in turn brings (12.) and (13.) under (8.). Hence our six conditions reduce to two, namely,

$$\alpha-\beta t\equiv 0 \ (s) \tag{8.}$$

$$\text{and } s\equiv 0 \ (\alpha^2-\beta^2 m) \tag{14.}$$

Moreover, since $\alpha-\beta t\equiv 0$ (s), we have $\alpha t-\beta t^2\equiv 0$ (s), and hence $\alpha^2-\beta^2 m\equiv 0$ (s). But this, together with (14.), gives us the equation

$$\pm s=\alpha^2-\beta^2 m \tag{15.}$$

Similarly for the case $m\equiv 1$ (4), we find that the conditions redvce to (8.) and the equation

$$\pm s=\alpha^2+\alpha\beta-\beta^2\frac{m-1}{4} \tag{16.}$$

Hence the necessary and sufficient condition that $(s,\ t+w)$ should be a principal ideal is that it should be possible to find rational integers α and β to satisfy equation (15.) or (16.) and congruence (8.).

If s is a prime number, the equation alone is sufficient.

As a special case, the conditions ars evidently fulfilled if the norm of $t+w$, t^2-w^2, is equal to s; in which case $(s,\ t+w)$ is the principal ideal corresponding to $t+w$.

Equation (15.), or (16.), gives the necessary condition for a principal ideal that $\left(\frac{\pm s}{m}\right)=1$, or $\left(\frac{\pm 4s}{m}\right)=1$, while for any ideal whatever we must have

$$\begin{cases} \left(\dfrac{m}{s}\right)=1. \\ \left(\dfrac{m}{4s}\right)=1. \end{cases}$$

The necessary and sufficient condition in the form to which we have reduced it is of little practical value as a test if $m>0$. For $m<0$ it may be useful.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

350. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Solve the equations:
$$\begin{aligned} &x+y+z=a_0,\\ &x+yu+zv=a_1,\\ &x+yu^2+zv^2=a_2,\\ &x+yu^3+zv^3=a_3,\\ &x+yu^4+zv^4=a_4. \end{aligned}$$

Solution by A. H. HOLMES, Brunswick, Maine, and the PROPOSER.

$x+y+z=a_0\ldots(1)$; $x+yu+zu=a_1\ldots(2)$; $x+yu^2+zv^2=a_2\ldots(3)$; $x+yu^3+zu^3=a_2\ldots(4)$; and $x+yu^4+zu^4=a_4\ldots(5)$.

Subtracting (1) from (2), (2) from (3), (3) from (4), and (4) from (5), we have,

$$\begin{aligned} &y(u-1)+z(v-1)=a_1-a_0\ldots(6);\\ &yu(u-1)+zu(v-1)=a_2-a_1\ldots(7);\\ &yu^2(u-1)+zv^2(v-1)=a_3-a_2\ldots(8);\text{ and}\\ &yu^3(u-1)+zv^3(v-1)=a_4-a_3\ldots(9). \end{aligned}$$

Eliminating y from (6), (7), (8), and (9), we have,

$$\begin{aligned} &(a_1-a_0)u-zu(v-1)=a_2-a_1-zv(v-1)\ldots(10),\\ &(a_2-a_1)u-zuv(v-1)=a_3-a_2-zv^2(v-1)\ldots(11),\text{ and}\\ &(a_3-a_2)u-zuv^2(v-1)=a_4-a_3-zv^3(v-1)\ldots(12). \end{aligned}$$

Eliminating z from (10), (11), and (12), we have,

$$\begin{aligned} &(a_1-a_0)uv-(a_2-a_1)v=(a_2-a_1)u-(a_3-a_2)\ldots(13),\text{ and}\\ &(a_2-a_1)uv-(a_3-a_2)v=(a_3-a_2)u-(a_4-a_3)\ldots(14). \end{aligned}$$

Eliminating v from (13) and (14), we have

$$\frac{(a_3-a_2)u-(a_4-a_3)}{(a_2-a_1)u-(a_3-a_2)}=\frac{(a_2-a_1)u-(a_3-a_2)}{(a_1-a_0)u-(a_2-a_1)}.$$

Putting $a_1-a_0=d_1$, $a_2-a_1=d_2$, $a_3-a_2=d_3$, $a_4-a_3=d_4$, and solving,

$$u=\frac{d_1d_4-d_2d_3\pm\sqrt{[(d_1d_4-d_2d_3)^2+4(d_1d_3-d_2^2)(d_2^2-d_2d_4)]}}{2(d_1d_3-d_2^2)}.$$

From (13) we find v. Then from (10) we find z. From (6), y, and from (1), x.

351. **Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.**

Solve, $y^2+yz+z^2=a^2\ldots(1)$.
$z^2+zx+x^2=b^2\ldots(2)$.
$x^2+xy+y^2=c^2\ldots(3)$.

I. **Solution by J. A. COLSON, Searsport, Maine.**

$b^2-c^2=(z-y)(x+y+z)$. $\therefore(b^2-c^2)x=(zx-xy)(x+y+z)$.

$c^2-a^2=(x-z)(x+y+z)$. $\therefore(c^2-a^2)y=(xy-yz)(x+y+z)$.

$a^2-b^2=(y-x)(x+y+z)$. $\therefore(a^2-b^2)z=(yz-zx)(x+y+z)$.

$\therefore(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z=0$.

For convenience, put $b^2-c^2=f$, $c^2-a^2=g$, and $a^2-b^2=h$. Then $f+g+h=0$, and $fx+gy+hz=0$.

$$\therefore z=-\frac{fx+gy}{h},\text{ and } x+y+z=x+y-\frac{fx+gy}{h}=\frac{(h-f)x+(h-g)y}{h}.$$

$$\therefore a^2-b^2=h=(y-x)(x+y+z)=(y-x)\frac{(h-f)x+(h-g)y}{h}.$$

$\therefore h^2=(f-h)x^2+(g-f)xy+(h-g)y^2$.

But from (3) we have $y^2=c^2-x^2-xy$.

Hence, $h^2=c^2(h-g)+(f+g-2h)x^2+(2g-f-h)xy=c^2(h-g)-3hx^2+3gxy$.

$$\therefore y=\frac{3hx^2+h^2+c^2(g-h)}{3gx}.$$

Substitute in (3), and we have

$$x^2+\frac{3hx^2+h^2+c^2(g-h)}{3g}+\frac{[3hx^2+h^2+c^2(g-h)]^2}{9g^2x^2}-c^2=0.$$

Hence, clearing of fractions and uniting, we have,

$$9(g^2+gh+h^2)x^4-3[c^2(2g^2-gh+2h^2)-h^2(g+2h)]x^2$$
$$+[h^2+c^2(g-h)]^2=0.$$

$\therefore 36(g^2+gh+h^2)^2x^4-12(g^2+gh+h^2)[c^2(2g^2-gh+2h^2)-h^2(g+2h)]x^2+[c^2(2g^2-gh+2h^2)-h^2(g+2h)]^2=[c^2(2g^2-gh+2h^2)-h^2(g+2h)]^2-4(g^2+gh+h^2)[h^2-c^2(g-h)]^2=9c^4g^2h^2-6c^2g^2h^2(2g+h)-3g^2h^4$.

$$\therefore 6(g^2+gh+h^2)x^2-[c^2(2g^2-gh+2h^2)-h^2(g+2h)]$$
$$=\pm gh\sqrt{[9c^4-6c^2(2g+h)-3h^2]}.$$

Giving g and h their original values, we have

$$6(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)x^2=2b^6+2c^6-a^6+4a^4(b^2+c^2)$$
$$-5a^2(b^4+c^4)-5a^2(b^4+c^4)+b^2c^2(b^2+c^2-3a^2)$$
$$\pm(a^2-b^2)(c^2-a^2)\sqrt{[3(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)]}.$$

If k=the area of a triangle whose sides are a, b, and c, then $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4=16k^2$.

Hence, $6(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)x^2$

$$=2b^6+2c^6-a^6+4a^4(b^2+c^2)-5a^2(b^4+c^4)$$
$$b^2c^2(b^2+c^2-3a^2)\pm4(a^2-b^2)(c^2-a^2)k\sqrt{3}.$$

Hence, by permuting the letters a, b, c we can find the values of y^2 and z^2 from the two following equations:

$$6(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)y^2=2c^6+2c^6+2a^6-b^6+4b^4(c^2+a^2)$$
$$-5b^2(c^4+a^4)+c^2a^2(c^2+a^2-3b^2)\pm4(b^2-c^2)(a^2-b^2)(k\sqrt{3}$$

and
$$6(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)z^2=2a^6+2b^6-c^6+4c^4(a^2+b^2)$$
$$-5c^2(a^4+b^4)+a^2b^2(a^2+b^2-3c^2)\pm4(c^2-a^2)(b^2-c^2)k\sqrt{3}.$$

II. Solution by **ARTEMAS MARTIN, LL. D.**, Editor and Publisher, Mathematical Magazine, Washington, D. C.

From the square of the sum of the given equations, subtract twice the sum of their squares and extract the square root of one-third of the remainder; then

$$xy+yz+xz=\pm\tfrac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^4+c^4)]}\ldots(4).$$

Subtracting twice (1) from the sum of (4) added to the sum of the given equations,

$$2x(x+y+z)=b^2+c^2-a^2\pm\tfrac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^4+c^4)]}\ldots(5).$$

Subtracting twice (2) and twice (3) in succession from the same sum,

$$2z(x+y+z)=a^2+c^2-b^2\pm\tfrac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^4+c^4)]}\ldots(6),$$
$$2z(x+y+c)=a^2+b^2-c^2\pm\tfrac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^2+c^4)]}\ldots(7).$$

Add the three equations (5), (6), and (7); then

$$2(x+y+z)(x+y+z)=2(x+y+z)^2=a^2+b^2+c^2$$
$$\pm\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^4+c^4)]}\ldots(8).$$

Extracting the square root of twice (8),

$$2(x+y+z)=\pm\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^4+c^4)]}\}}\ldots(9).$$

Dividing (5), (6), and (7) in succession by (9),

$$x=\frac{b^2+c^2-a^2\pm\frac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}\}}},$$

$$y=\frac{a^2+c^2-b^2\pm\frac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}\}}},$$

$$z=\frac{a^2+b^2-c^2\pm\frac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}\}}}.$$

Also solved by A. H. Holmes, J. Scheffer, and V. M. Spunar.

For a number of different solutions of this problem, when the known quantities are not squared, see *The Mathematical Magazine*, published by Dr. Artemas Martin, Vol. II, pp. 141-144, and pp. 193-196. ED. F.

GEOMETRY.

375. Proposed by C. N. SCHMALL, New York City.

From a point P on a circle there are drawn three chords PA, PB, PC. Show that the circles described on these chords as diameters intersect again in three collinear points.

I. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology.

Take the point P as the origin of polar coördinates, and the diameter through P as the initial line. The coördinates of the points A, B, and C are, respectively, $2a\cos\alpha$, α; $2a\cos\beta$, β; $2a\cos\gamma$, γ; α, β, and γ being the vectorial angles.

The equations of the circles described upon the chords as diameters will be

$$\rho=2a\cos\alpha\cos(\theta-\alpha),$$
$$\rho=2a\cos\beta\cos(\theta-\beta),$$
$$\rho=2a\cos\gamma\cos(\theta-\gamma),$$

whence the coördinates of the points of intersection are

$$2a\cos\beta\cos\gamma,\ \beta+\gamma;\quad 2a\cos\gamma\cos\alpha,\ \gamma+\alpha;\quad 2a\cos\alpha\cos\beta,\ \alpha+\beta.$$

These points are all on the straight line whose equation is

$$2a\cos\alpha\cos\beta\cos\gamma=\rho\cos(\theta-\alpha-\beta-\gamma).$$

Join the points A, B, and C to make a triangle. The points of intersection are then the feet of the perpendiculars let fall from P upon the three sides, and the line through the points of intersection is the pedal line of P with respect to the triangle.

II. Solution by S. LEFSEHETZ, Clark University.

If we transform by inversion, the pole of inversion being in P, the transformed of the three circles of diameters PA, PB, and PC are perpendiculars at PA, PB, and PC in A', B', and C', points where these three lines meet the line obtained by transformation of the given circle. These three perpendiculars envelop a parabola of focus P; therefore, the circle circumscribed to the triangle they form passes through P, — a well known property of the parabola. By transforming back, we obtain a straight line and the proposition is thus proved.

Also solved by the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

355. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Solve the equations:

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}=a_1;$$
$$\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=a_2;$$
$$\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}=a_3;$$
$$\cdots\cdots\cdots$$
$$\frac{1}{x_{n-2}}+\frac{1}{x_{n-1}}+\frac{1}{x_n}=a_{n-2};$$
$$\frac{1}{x_{n-1}}+\frac{1}{x_n}+\frac{1}{x_1}=a_{n-1};$$
$$\frac{1}{x_n}+\frac{1}{x_1}+\frac{1}{x_2}=a_n.$$

356. Proposed by ARTEMAS MARTIN, Ph. D., Washington, D. C.

Solve by quadratics, if possible, the equations,

$$w(x+y+z)=a,\quad x(w+y+z)=b,$$
$$y(w+x+z)=c,\quad z(w+x+y)=d.$$

[From the *Mathematical Magazine*, Vol. II, p. 256.]

GEOMETRY.

387. **Proposed by DANIEL KRETH, Oxford, Iowa.**

A lot 100 feet long and 60 feet wide, has a walk extending from one corner half way around it, and occupying one-third of the area. Required the width of the walk. A geometrical construction is desired.

388. **Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

A conic is inscribed in a triangle and one focus lies on the polar circle of the triangle. Prove that the corresponding directrix passes through the center of perpendiculars.

389. **Proposed by H. PRIME, Boston, Mass.**

On the same side of a given base, triangles are erected such that the bisectors of their vertex angles all pass through a given point. Find the locus of the vertices (i) when the vertex angle are all equal, (ii) when the vertex angles are all unequal.

CALCULUS.

311. **Proposed by WILMER THOMPSON, Senior, Drury College.**

Solve the differential equation,

$$\left(\frac{dy}{dx}\right)^3+x^3=ax\left(\frac{dy}{dx}\right).$$

[From Forsythe's *Differential Equations*, p. 47.]

312. **Proposed by C. N. SCHMALL, New York City.**

Given $y^3-3y+x=0$, prove by Maclaurin's theorem, that

$$y=\frac{x}{3}+\frac{x^3}{3^4}+\frac{x^5}{3^6}+\text{etc.}$$

MECHANICS.

261. **Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.**

A man six feet high, walking at a rate of 100 yards a minute, crosses a muddy road close behind a wheel of a carriage which is going thrice as fast and in a direction at right angles to that of the man's motion. The diameter of the wheel is five feet. If, when the man is four feet from the middle of the wheel the mud is splashed up to the hight of seven feet, will any of it touch him? Unsolved in *Educational Times.*

262. **Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.**

A hemispherical shell, whose radius is equal to the mean radius of the earth and whose thickness is one centimeter, is constructed of a matter whose density is equal to the mean density of the earth. A particle starts from rest at the center of the shell under the action of the attraction of the shell. Express as the decimal of a year the time it takes the particle to reach the surface of the shell, and find the velocity in centimeters per second of the particle just before it reaches the shell. Unsolved in *Educational Times.*

NOTES AND NEWS.

Problems for solution in the several departments from various contributors are desired. Please send us good problems for solution and do not let a few do all the proposing of problems. F.

We learn, from *Science*, that Professor Arthur Ray Maxson, instructor in mathematics at Columbia University, died April 13, at the age of thirty years. Professor Maxson was for several years a valuable contributor to the MONTHLY, and we shall miss his loyalty and support. F.

We are receiving numerous complaints from subscribers who fail to get their numbers of the MONTHLY. Our only answer is that we mail a copy to each of our subscribers, and that the failure to receive them must lie in the great majority of cases with the Postal Department. We hope our subscribers will bear patiently with us in this matter until the Department gets through punishing the big magazines and political delinquents for their unfriendly criticisms, after which it is hoped some attention will be given to efficiency and public service. The MONTHLY is mailed on the 28th of each month. F.

The last number of the *L'Enseignement Mathêmatique*, March, 1911, page 148, contains some details in regard to the *Encyclopedia of elementary mathematics* which the Italian mathematical society "Mathesis" has had under consideration for more than a year. The work will contain forty-four monographs under the following headings: Logic of mathematics; elementary arithmetic; theory of numbers; the notion of number and its extensions; limits, series, continued fractions, and infinite products; progressions and logarithms; literal calculus and algebraic identities; combinatory analysis, determinants, and linear equations; equations whose degree exceeds one; algebraic problems and their discussion; elements of the infinitesimal calculus; relations between analysis and elementary algebra; elementary properties of plane and space figures; theory of measure and applications; geometry of the triangle and of the tetrahedron; regular polygons and regular polyhedrons; elementary geometric transformations; linear systems of circles and spheres; geometry on the sphere; sections of the cylinder and the circular cone; maxima and minima in geometry; methods of solution of geometric problems and classic problems; foundations of elementary geometry; circular and hyperbolic functions, and plane and spherical trigonometry; vectorial calculus; elements of analytic geometry; elements of projective geometry; elements of descriptive geometry; special curves and surfaces; non-euclidean geometry; non-archimedean geometry; geometric representation of complex numbers; relations between elementary geometry and the theories of higher geometry; units of measure; numerical approximations and graphic calculus; calculus of probability and the theory of errors;

elementary applications of mathematics to the physical sciences; mathematics of statistics; mathematics of finance; history of elementary mathematics; didactic methods and text-books; mathematical recreations; instruments; models.

From these headings, and from the standing of the men who have undertaken the preparation of these monographs, it appears that Italy will soon have a work which will be extremely useful to her teachers of secondary mathematics. It is to be hoped that its appearance will tend to attract attention to the great need of an extensive mathematical encyclopedia in English, dealing with secondary mathematics. Those whose interests are mainly in higher mathematics are generally in position to use the great encyclopedia which is now being published in French and German, but teachers of secondary mathematics cannot usually derive so much profit from this great work. It would appear that our need of an encyclopedia of elementary mathematics should be greater than the need in Italy, and that this need should receive attention at our teachers meetings, especially at those having a national scope. M.

BOOKS.

Introduction A La Théorie de Nombres Algébriques. Par Dr. J. Sommer, Professeur A La "Technische Hochschule" de Dantzig. Edition Francaise Revue et Augmenteé. Traduit de L'Allemend Par A. Lévy, Professeur Au Lycée Saint-Louis. Avec Préface de J. Hadamard, Professeur Au Collége de France. Paper Cover, x+376 pages. Price, 15 francs. Paris: A. Hermann et Fils.

This work is divided into five chapters. The first chapter is the Introduction, treating the theorems of ordinary number theory. Chapter II deals with quadratic bodies; Chapter III, applications of the theory of quadratic bodies; Chapter IV, bodies of the third degree; and Chapter V, relative bodies. The treatment of the various topics indicated in the above enumeration is very clear and free from unnecessary complex and abstruce statements. In Chapter III, pages 184-201, we find a discussion of the last theorem of Fermat. We especially recommend a careful reading of this discussion. It will clear up a number of difficulties connected with this famous problem, interest in which has recently been revived by an offer of 100,000 marks for a proof or disproof of its truth. F.

Théorie des Corps de Nombres Algébriques. Par David Hilbert, Professeur A L'Université de Goettingen. Ouvrage traduit de L'Allemand Par A. Lévy, Professeur Aut Lycée Saint-Louis. Première Partie Le Nombre Algébriques et la Corps Algébriques. Paper Cover, 72 pages. Price, 3 francs. Paris: A. Hermann et Fils.

This translation places the profound researches of Hilbert on the theory of ideals in French garb, and enables English readers to read them both in the original and in the French. F.

A Text-book of Differential Calculus, with Numerous Worked Out Examples. By Ganesh Prasad, B. A. (Cantab), D. Sc. (Allahabad). 8vo. Cloth, xii+161 pages. Price, $1.50. New York and London: Longmans, Green & Co.

This work aims to establish the fundamental principles of the Differential Calculus on a firm foundation and to present its first principles so clearly as to be readily understood by the beginner. The book closes with a few miscellaneous notes among which is a discussion of Weierstrass's Function. F.

The Technical World Magazine for March contains among other articles of interest, one on the Story of the Speaking Dog. The actual breaking into human speech by this animal is one of the most extraordinary things of modern times. F.

The American Journal of Mathematics for April contains the following articles: On Three-Spreads Satisfying Four or More Homogeneous Linear Partial Differential Equations of the Second Order, by Charles H. Sisam; Some Properties of Lines in Space of Four Dimensions and Their Interpretation in the Geometry of the Circle in Space of Three Dimensions, by C. L. E. Moore; On the Geometry of Line Elements in the Plane with Reference to Oscillating Circles, by George F. Gundelfinger; Binary Modular Groups and Their Invariants, by Leonard E. Dickson; The Group of Turns and Slides and the Geometry of Turbines, by Edward Kasner. F.

The Transactions of the American Mathematical Society for April contains the following articles: Biorthogonal Systems of Functions, by Anna J. Pell; Applications of Biorthogonal Systems of Functions to the Theory of Integral Equations, by Anna J. Pell; On the Uniform Convergence of the Developments in Bessel Functions, by C. N. Moore; Determination of the Ordinary and Modular Ternary Linear Groups, by H. H. Mitchell; General Theory of Linear Differential Equations, by Geo. D. Birkhoff. F.

The Mathematical Magazine. A Journal Devoted to Elementary Mathematics. Edited and Published by Artemas Martin, Ph. D., LL. D., Washington, D. C. Issued Quarterly. Price, $1.00 per year, in advance.

The last number, viz., No. 12, of Vol. II, recently published, contains the following articles: About Sixth Power Numbers whose Sum is a Sixth Power, by Artemas Martin; Elementary Proof of Properties of Numbers, by Rollin A. Harris; The United States Sinking-Fund, by Theodore L. DeLand; On Rational Scalene Triangles, by Artemas Martin; On Biquadrate Numbers, by Benton Haldeman; On Rational Right-Angled Triangles, by Artemas Martin.

It is to be regretted that Dr. Martin does not have the leisure to publish the magazine regularly. *The Mathematical Magazine* and *The Mathematical Visitor*, both edited and published by Dr. Martin, even the type-setting being done by himself, are the finest specimens of the type-setters' art in mathematical typography. F.

The Problem of the Angle-Bisectors. A Dissertation Submitted to the Ogden Graduate School of Science of the University of Chicago in Candidacy for the Degree of Doctor of Philosophy, by Richard Philip Baker. 98 pages. Price, $1.10. The University of Chicago Press.

This Thesis is a very detailed discussion of the famous problem, "To construct a triangle, having given the three angle bisectors." This problem, which might easily be attacked by a freshman with the hope of success, proves to be a very intractable one, and not amenable to the ordinary processes of Euclidean Geometry. Dr. Baker has performed a very creditable piece of work in the production of this interesting thesis. F.

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NUMBER.

By LOUIS C. KARPINSKI, University of Michigan.

Some fundamental errors in regard to the nature and origin of number have been given such wide circulation in recent works on psychology and the psychology of number that it seems desirable to present views which are based upon the scientific and revolutionary work upon the nature of the number idea which has been done by Cantor,* Dedekind, Peano, Frege, and others. Psychologists, philosophers, and pedagogs have treated the subject of the definition of number and the genesis of the number idea in apparent ignorance of the fact that great mathematicians have also labored in this field. While some may dispute the right of mathematicians to discuss so psychological a matter as the genesis of the number idea, no one can dispute that a somewhat logical definition of number is a necessary basis for work on the genesis of the number idea.

One faulty definition which persists is that number is ratio. If that be so, pray what is ratio? Surely the idea of ratio is no more evident than is that of number.

Upon this poor foundation a method of teaching arithmetic is based whose merit, if any, is in spite of this fundamentally worthless concept of number. To confuse number with ratio is to confuse number with an application of number. Logically ratio is an unsymmetric relation between two numbers.

Some writers would have us believe that number takes its origin in counting. Here, again, we may ask, is counting intuitive? Is the idea of counting any simpler than the idea of number? Counting presupposes the presence of the number idea in the mind of the counter. Children often go through the counting process in much the same way that they repeat Old Mother Goose jingles; as much number concept is present in one operation as in the other. The same criticism holds against that view of number

* Cantor, Georg, *Mathematische Annalen*, Vol. XLVI, p. 481, *Beitrage zur Begründung transfiniten Mengenlehre*. Dedekind, R. *Was sind und was sollen die Zahlen?* Also translated into English by W. W. Beman in *Essays on the Theory of Numbers*, Open Court Co. Frege, G. *Grundlagen der Arithmetic.*

which regards the number idea as evolved in some mysterious way by successive strokes of attention.

The need for measuring is given by a very eminent philosopher as the determining factor which evolves number. Necessity may be the mother of invention in a general way, but certainly not of the invention of the idea of number. The slightest reflection shows that measuring involves having number concepts as a basis.

It would seem out of place to refer to those writers who regard numbers as objects. However, there yet exists schools in America and Germany in which the children ostensibly learn all about the number four before proceeding to the number five.

In all of these we have the applications of number confounded with the actual number idea. Most eminent mathematicians and philosophers have made such errors. Euler defined number as ratio, and the substance of the definition by Leibnitz is that number is obtained by comparing a given magnitude with a unit magnitude of the same kind. However, neither did these men nor did any of the great mathematicians of more than one hundred years ago occupy themselves at all deeply with the definition of number. They assumed number as being *a priori* and a knowledge of the fundamental operations with integers and fractions as presupposable. Within the last fifty years the philosophical bases of the science of geometry and analysis have been carefully studied. In geometry out of these studies came the non-Euclidean geometries. In analysis the result has been to show that logic and mathematics are fundamentally sister studies, in fact almost Siamese twins.

A logical definition of cardinal number presupposes the idea of a group or class of objects and the notion of membership of a group. It also involves the idea of one to one correspondence; that is, the definition implies that we know what is meant when it is stated that each child has an orange, "to each child an orange." Groups of objects are in one to one correspondence when to each object of one group there corresponds one and only one object of the other group, and vice versa. Upon the basis of these ideas we may say that that concept which is common to all groupes of objects which may be placed in one to one correspondence with the objects of any given group is called the number of the given group. This definition is essentially the same as stating that the number of a group of objects is that concept which remains when we abstract from the particular objects which make up the group any special characteristics of those objects as well as any notion of order.*

It is not to be denied that logical objections may be made to these definitions, but the advance is so great over the older ideas that any modern attempt at a psychology of number must take account of this work.

* These two definitions are in substance given by G. Cantor in the article noted above. Dedekind and Frege also use the one-to-one correspondence idea.

Preceding the acquisition of definite number concepts is the acquisition of the ideas involved by the words, "another," and "more," used quantitatively and also as equivalent of "others," and the perception of groups of like objects as groups. The group idea is constantly brought to the child's attention by the use of the plural—apples, pears, balls, etc. This forms the important step in the evolution of the number idea, of perceiving likeness in different objects and of naming that quality.

At some stage in the child's development, according to Perez and others usually about the age of two years, the child is ready to acquire a name for that special group characteristic which is constantly presented to him in his own hands, feet, eyes, and ears. The child delights in the word two as emphasizing a new idea and seeks application for it. Two apples, two oranges, two pears, two ladies, are combinations that readily follow upon the presentation of the objects. Even yet there may be confusion of groups of two with groups of greater number, but the great step is made of a separate name to denote that property of combination of like objects into one group.

The child experiences a sense of great pleasure in the acquisition of this new idea. The feeling of pleasure is analagous to that experienced by an adult on attaining a new mental experience such as going up in a balloon or descending into a large underground cave. Pierre Loti refers to a similar feeling of exhiliration which he experienced as a young child on discovering that he could jump.

At about this point the child's number system consists of object, two, many. The base of this number system is the number two. Primitive people exist who have not attained any farther than this number system. The binary system of numbering is almost universal among the tribes of Aborigines in Australia* and is common among the tribes of South America. These tribes have words for one, two. Three is given as two and one; four as two and two; five as two, two, one, or sometimes one, two, two, and six as two, two, two. Usually the system does not go much farther than this.

Following two and many as more than two, or possibly almost accompanying these ideas, is the idea of unity. While logically the definition of unity rests on identity, psychologically the concept of unity is that of membership of a group. The child does not easily speak of "one" mother, whereas "one apple" is quite easily achieved.

Three follows very readily as two and one,† and four as two-two. There is no logical nor psychological reason why "three" need appear before "four," nor "two" before "three" in the child mind. Indeed, if the groups of objects most constantly brought before the child's attention by his own

---

* Mathew, John, *Eaglehawk and Crow, a Study of Australian Aborigines.*

† In this connection it is striking that the root of three, "tri," signifies "more than," meaning one more than the two preceding numbers. Given by Bopp, *Grammaire comparée.*

body were groups of threes, that would be the first fundamental group, and three would be the first number appreciated.

Experimental psychologists find that groups of one, two, three or four objects are immediately perceived as to number, that is, that the time required to recognize that there are four objects in a group is not apparently longer than that there are two or three, whereas to group five or six objects takes a longer time unless these objects are favorably arranged. In this psychological fact we may have the reason why many primitive people do not get beyond the number four. We can only conjecture that there is some connection between this and the fact that we unconsciously have a four group, our fingers, constantly thrust before us.

Five comes as a second natural number base. It is either the base or subsidiary base of practically all number systems beyond the binary stage. The Roman Numerals are striking illustrations of the use of five as a secondary base, as the system is a ten system. The same is true of the Attic system or numerals in use for many centuries among the Greeks. In this system

1=I	5=Γ
10=Δ	50=$\Gamma\Delta$
100=H	500=Γ_{H}
1000=X	5000=Γ_{X}
10000=M	50000=Γ_{M}

Among the Mayas who have a twenty system the five also comes in as a subsidiary base. Possibly the rhythm may account for the appearance of five and multiples of five in so many of the children's counting games. In *Pädagogische Studien* Dr. E. Wilk has emphasized the fundamental importance of the five system in a series of articles on a new number method based on the natural origin of number and reckoning. Dr. Wilk states: "It is the most important result of my investigation concerning the origin of numbers that these can be formed only by the introduction of a number system." To this end he names the use of the fingers as the best material for the early work. In our schools we have undoubtedly gone too far in banishing the fingers from the early number work, but best material for number work does not exist just as best food for children does not exist. Especially with weak-minded children and children slow to grasp number facts the finger reckoning is of vital importance. Attention has recently been called to this in the *Zietschrift für Kinder forschung* (with special reference to pedagogical pathology) by Dr. H. Noll in an article on finger activity and finger reckoning as aids in developing the intelligence and the reckoning ability in weak-minded children.

Six, seven, eight and nine come as five and one, five and two, five and three, five and four. No difficulty attaches to developing ten as a new unit

as the child's ten fingers stare him in the face and force ten as a unit. Our system of money also greatly facilitates the acquisition of ten as a unit, as most children have all too intimate acquaintance with pennies and nickles and dimes. Probably, too, the money forms as convenient material as any, and more impressive than most, for developing the ideas of eleven, twelve, up to nineteen, and for twenty, thirty, forty, etc. The universal ten system among civilized peoples is due, as Aristotle first pointed out, to the fact that we have ten fingers. Certain mathematicians and even psychologists have argued that a twelve system would be better adapted to human needs, and some feeble attempts have been made to institute such a system. Logically and mathematically twelve would be a better system for adults, as the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ would be .6, .4 and .3 respectively, meaning $\frac{6}{12}$, $\frac{4}{12}$ and $\frac{3}{12}$. Unfortunately we are not logical beings but psychological, bound hand and foot (by fingers and toes) to the decimal system. Whether this influence of the fingers is subconscious or conscious, cannot be definitely shown, but the number systems of all civilised races show that this influence is the most powerful factor in the forming of a number system.

To summarize, the steps in the development of definite number ideas would seem to be as follows:

(1). Many, another, more. Perception of likeness in other individuals, also noting of plurals.

(2). Two.

(3). Many, as more than two.

(4). Two and one. Two as a number base.

(5). Two and two.

(6). Five. as two, two and one, or four and one. Five as a number base.

(7). Five and one, five and two, five and three, five and four.

(8). Ten as two fives, a new unit.

(9). 11-20 as ten and one up to two tens.

(10). 20, 30, 40, 50, 60, etc., as 2, 3, 4, 5, ... 10 tenf.

The application of these ideas to the teaching of arithmetic is immediate. In the earliest number work it means continued emphasis on fundamental group ideas. Two will always be associated with the hands, four with the fingers without the thumb, five with the fingers of one hand, seven with the days of the week, ten with the fingers of two hands, and other numbers with characteristic groups in the school room, *e. g.*, if there happen to be six panes of glass in the window, six will be associated with the number of window panes. The earliest number concepts do not come from the consideration of mathematical objects such as cubes and squares nor even splints. By presenting these first the development of the number idea is retarded as the child mind is required to struggle with these unfamiliar objects. To the child these geometrical objects have no meaning; he does not see why he should observe them. Much easier is it for him to get the

number phase from objects with which he is familiar. In fingers and apples he has an interest and any new ideas about these old friends are seized with some avidity.

To question the child's power to abstract the difference of like objects is to question his power of imagination. that prime requisite of a mathematician which power the child has most preeminently.

For the further work in arithmetic this development means the constant and recurring emphasis of the decimal system. Eighteen hundred years ago Nikomachus of Gerasa, the father of Arithmetic, did better than we in giving a table of multiples only up to 9×9, and the earliest printed arithmetics used the ordinary multiplication table only to 9×9 or 10×10. By so doing is emphasized the use of the decimal system, as all further multiplicative combinations involve only these fundamental number of facts. With the addition tables, too, there needs to be emphasis on the system. Here we may see one great way to simplify our arithmetic, namely, to play the system.

A SOLUTION OF THE BIQUADRATIC EQUATION.

By LEROY A. HOWLAND, Middletown, Connecticut.

The following solution of the biquadratic rests upon the analytic operations suggested by geometrical considerations. No assumption is made as to the form of the solution. Each step in the solution and in the discussion of multiple roots has its geometrical analogon. The cubic resolvent has a geometrical meaning and its discriminant is at the same time the discriminant of the biquadratic. M. Fritz Hofmann has used the degenerate members of a family of conics through four points to determine the roots of the biquadratic,* but in an entirely different way. He does not appear to have extended his method to a discussion of the nature of the roots.

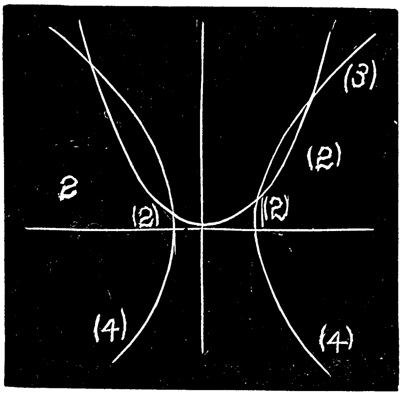


Fig. 1.

1. Solution. Let the biquadratic be

$$a_0z^4+4a_1z^3+6a_2z^2+4a_3z+a_4=0 \quad (a\neq 0) \tag{1}$$

This goes over by the substitution $z=\dfrac{x-a_1}{a_0}$ into

$$x^4+ax^2+bx+c=0, \tag{2}$$

* *Nouvelles Annales de Mathématiques, Troisième Série, 7.*

where $a=6(a_0a_2-a_1^2)$,
$b=4(a_3^2-3a_0a_1a_2+2a_1^3)$,
$c=a_0^3a_4-4a_0^2a_1a_3+6a_0a_1^2a_2-3a_1^4$.

(2) represents four straight lines parallel to the Y axis. These cut the parabola

$$(3) \qquad y=x^2$$

in four finite points. The conic

$$(4) \qquad ax^2+y^2+bx+c=0$$

cuts (3) in these same four points (cf. Fig. 1). A conic through the intersection of (3) and (4) is

$$(5) \qquad (a-\lambda)x^2+y^2+bx+\lambda y+c=0.$$

This degenerates if

$$-D=\begin{vmatrix} a-\lambda & 0 & b/2 \\ 0 & 1 & \lambda/2 \\ b/2 & \lambda/2 & c \end{vmatrix}=0$$

or, (6) $$\lambda^3-a\lambda^2-4c\lambda+4ac-b^2=0.$$

The solution of (6) gives three pairs of straight lines, each of which intersects the others in the intersections of (2) and (3). The solution of the biquadratic is then accomplished by the solution of (6) and certain pairs of simultaneous *linear* equations (cf. Fig. 2).

Solving (5) for y, we have

$$2y=-\lambda\pm\sqrt{[4(\lambda-a)x^2-4bx+\lambda^2-4c]}.$$

Using (6), this gives

$$2y=-\lambda\pm[2\sqrt{(\lambda-a)}x-\frac{b}{\sqrt{(\lambda-a)}}].$$

If λ_j and λ_k are two roots of (6) we have the following values for x,

$$
(7) \qquad \begin{array}{c} \frac{1}{2}\left[\pm \frac{\lambda_j-\lambda_k}{\mu_i+\mu_k} + \frac{b}{\mu_j\,\mu_k} \right] \\ \\ \frac{1}{2}\left[\pm \frac{\lambda_j-\lambda_k}{\mu_j-\mu_k} - \frac{b}{\mu_j\,\mu_k} \right] \end{array}
$$

where $\mu_i = \sqrt{(\lambda_i - a)}$.

2. Multiple Roots.

Fig. 2 represents the general case. Fig. 3 shows the special cases which may occur.

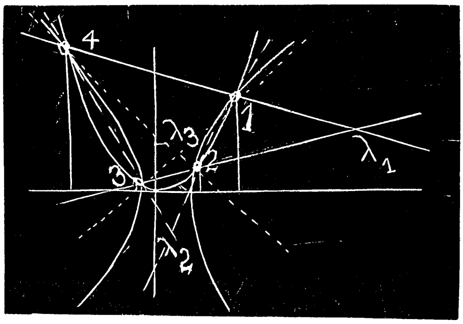


Fig. 2.

In cases I and II, two pairs of lines become coincident. Hence (6) must have a double root.

In case III, since the points approach coincidence along the parabola, each of the lines 12, 23, and 31 approaches the tangent to the parabola at the point of coincidence. Hence equation (6) must have a triple root.

Similar argument shows that in case IV all six lines coincide. We shall now express these conditions analytically.

I. The condition for a double root of (6) is

$$
\begin{aligned}
&G^2 + 4H^3 = 0 \\
\text{where } &G = \tfrac{1}{27}(72ac - 2a^3 - 27b^2), \\
&H = -\tfrac{1}{9}(a^2 + 12c),
\end{aligned}
$$

or in terms of the coefficients of (1),

$$
\begin{aligned}
G &= 16a_0^3(a_0a_2a_4 - a_0a_3^2 + 2a_1a_2a_3 - a_1^2a_4 - a_2^3), \\
H &= -4/3a_0^2(a_0a_4 - 4a_1a_3 + 3a_2^2).
\end{aligned}
$$

The quantities in parentheses are denoted by J and I, respectively. Hence, $G^2 + 4H^3 = -256/27a_0^3(I^3 - 27J^2)$, and the necessary and sufficient condition for a double root of (1) is

$$
(8) \qquad \triangle \equiv I^3 - 27J^2 = 0.
$$

II. For two double roots, the lines corresponding to the double root of (6) must be coincident. (5) will represent parallel lines if $D = 0$ and $\lambda - a = 0$.

We see by substitution that a necessary condition that $\lambda=a$ be a solution of (6) is

(9) $$b\equiv a_0^2 a_3-3a_0 a_1 a_2+2a_1^3=0.$$

Equation (5) then becomes

$$y^2+ay+c=0,$$

and we obtain the further necessary condition

(10) $$a^2-4c=0.$$

These conditions, (9) and (10), are evidently also sufficient.*

III. A triple root of (6) is neeessary and sufficient for a triple root of (1). The condition for this is $G=H=0$, or what is the same,

(11) $$I=J=0.$$

IV. For a quadruple root we must have, in addition to (11), the conics degenerate into coincident lines. This gives, as before, $\lambda=a$, the triple root of (6), and hence $b=0$. It then follows that $a=c=0$. These conditions are sufficient, for if they are fulfilled, (2) becomes $x^4=0$, and (1) becomes

$$\frac{1}{a_0^3}(a_0 z+a_1)^4=0.$$

In terms of the coefficients of (1) the conditions may be written

(12) $$\frac{a_0}{a_1}=\frac{a_1}{a_2}=\frac{a_2}{a_3}=\frac{a_3}{a_4}$$

unless one of the quantities a_1, a_2, a_3, a_4 is zero, in which case they are all zero and (1) reduces to $a_0 z^4=0$.

3. Reality of Roots.

I. If $G^2-4H^3>0$ or $\triangle<0$, the equation (6) has two imaginary roots, $\alpha\pm i\beta$.

$$\mu_j=\sqrt{(\alpha-a+i\beta)},\quad \mu_k=\sqrt{(\alpha-a-i\beta)},$$
$$(\mu_j\,\mu_k)^2=(\alpha-a)^2+\beta^2,$$
$$(\mu_j\pm\mu_k)^2=2(\alpha-a)\pm2\sqrt{[(\alpha-a)^2+\beta^2]},$$
$$\lambda_j-\lambda_k=2i\beta.$$

* We find the statement sometimes made that (9) and (10) along with the condition $\triangle=0$ are equivalent to two independent conditions only. If this means merely that $\triangle=0$ is a consequence of (9) and (10) the statement is correct. It is not true that $\triangle=0$ with either (9) or (10) forms a set of sufficient conditions for two double roots.

Hence it appears that $\mu_j \mu_k$ and $\mu_j+\mu_k$ are real, $\mu_j-\mu_k$ and $\lambda_j-\lambda_k$ are pure imaginary, and two values of x are real and two complex.

II. Before proceeding to the general cases $\triangle \gtreqq 0$, we shall take up the special case where $b=0$. (2) becomes $x^4+ax^2+c=0$, whence $x= \pm\sqrt{\dfrac{-a \pm \sqrt{(a^2-4c)}}{2}}$, $\triangle$ becomes $\dfrac{c(a^2-4c)^2}{16a_0^3}$, and we have two sub-cases.

1) $\triangle>0$, hence $c>0$.

In order for x to be real the radicand above must be real and positive. For this it is necessary and sufficient that $a^2-4c>0$ and $a<0$. All values of x are then real, otherwise they are all complex.

2) $\triangle=0$, hence either, a) $c=0$, or b) $a^2-4c=0$.

For a), (2) becomes $x^4+ax^2=0$, and has a real double root and two other roots which are real or complex according as $a\leqq 0$ or $a>0$.
For b), (2) becomes $(x^2+\frac{1}{2}a)^2=0$, and has two double roots, both real or both complex under the same conditions as in a.

III. We shall now assume $b\neq 0$ and consider the otherwise general case $\triangle>0$.

If $G^2+4H^3<0$ or $\triangle>0$ the roots of (6) are all real. In this case it is evident from the form of x in (7) that all values of x are real if all roots of (6) are greater than a and all complex if two roots of (6) are less than a. We will now apply the theorem of Fourier to see where the roots of (6) lie.

Let $f(\lambda)=\lambda^3-a\lambda^2-4c\lambda+4ac-b^2$,

$f'(\lambda)=3\lambda^2-2a\lambda-4c$,

$f''(\lambda)=6\lambda-2a$,

$f'''(\lambda)=6$.

Whence $f(a)=-b^2$, $f'(a)=a^2-4c$, $f''(a)=4a$, $f'''(a)=6$. This suggests the following division:

1) $a^2-4c>0 \quad a>0$,	4) $a^2-4c<0 \quad a<0$,
2) $a^2-4c>0 \quad a<0$,	5) $a^2-4c<0 \quad a=0$,
3) $a^2-4c<0 \quad a>0$,	6) $a^2-4c=0 \quad a>0$.

The other possibilities are $a^2-4c>0$, $a=0$; $a^2-4c=0$, $a<0$; $a^2-4c=0$, $a=0$;

but these are all ruled out, for G^2+4H^3 reduces to $b^4-\frac{64}{27}c^3$; $b^2(b^2-\frac{32}{27}a^3)$; b^4 in the three cases, respectively, and can in no case be negative. We have the following table:

	$\lambda=a$						$\lambda=\infty$	$\lambda=-\infty$
	1	2	3	4	5	6		
$f(\lambda)$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$
$f'(\lambda)$	$+$	$+$	$-$	$-$	$-$	0	$+$	$+$
$f''(\lambda)$	$+$	$-$	$+$	$-$	0	$+$	$+$	$-$
$f'''(\lambda)$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$

It appears then in case (2) that all roots are greater than a, while in every other case only one root is greater than a. In the case of real distinct roots of (6) then the necessary and sufficient condition that all values of x be real is $a<0$, $a^2-4c>0$. If these conditions are not fulfilled the values of x are all complex.

IV. $\triangle=0$, with the further assumption that $b\neq 0$. Let $\lambda_j=\lambda$ be the simple root of (6). It is greater than a and in case the double root is also greater than a, all values of x are real. The condition for this is easily seen to be as in case III, $a<0$, $a^2>4c$. Putting $2x$ in the form

$$\pm\sqrt{(\lambda-a)}+\frac{\mp\sqrt{(\lambda-a)(\lambda_k-a)}+b}{\sqrt{(\lambda-a)}\sqrt{(\lambda_k-a)}},\quad \pm\sqrt{(\lambda-a)}+\frac{\pm\sqrt{(\lambda-a)(\lambda_k-a)}-b}{\sqrt{(\lambda-a)}\sqrt{(\lambda_k-a)}},$$

we see that in all other cases, since $\lambda-a>0$ while $\lambda_k-a<0$, we can have real values of x only when

$$\sqrt{(\lambda-a)(\lambda_k-a)}=\pm b.$$

Since the sum of the three roots of (6) is a, we have $\lambda_k=\dfrac{a-\lambda}{2}$, and we can transform the condition above into

$$(13)\qquad \begin{cases}(\lambda+a)\sqrt{(\lambda-a)}=+2b, \text{ or}\\(\lambda+a)\sqrt{(\lambda-a)}=-2b.\end{cases}$$

We can have real values of x then only if

$$(14)\qquad (\lambda+a)^2(\lambda-a)=-4b^2.$$

The resultant of (6) and (14) is $\triangle$. This is, of course, to be foreseen, since whenever $\triangle=0$ there are two equal values and hence always two real values.

(The case where there are two double roots has been excluded because $b \neq 0$.) The conditions (13) are both satisfied only in case $b=0$, and this we have already considered.

We will summarize the results of this section.

1) $\triangle < 0$, two roots real, two complex.

2) $\triangle > 0$,

   a) $a<0$, $a^2-4c>0$, all roots real.

   b) Otherwise all roots complex.

3) $\triangle = 0$,*

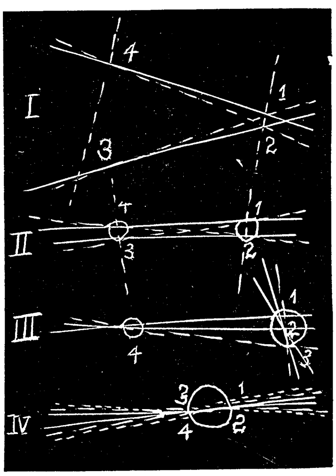
   a) $a<0$, $a^2-4c>0$, all roots real.

   b) Otherwise two real and two complex, except in the following cases:

   c) $a>0$, $b=0$, $a^2-4c=0$, two double complex roots.

   d) $a<0$, $b=0$, $a^2-4c=0$, two double real roots.

   e) $a=b=c=0$, all roots real.



Double root.

2 Double roots.

Triple root.

Quadruple root.

Fig. 3.

* In many standard works this case is not considered. It is discussed by Heinrich Weber in his *Lehrbuch der Algebra*, but the discussion contains an error. [Weber noticed this error himself in his *Berichtigungen* at the end of the volume.] One finds on page 278 of volume I, second edition, the statement: Die biquadratische Gleichung hat zwei Paare gleicher, reeller Wurzeln, wenn $D=0$, $a<0$, $a^2-4c=0$ und zwei Paare gleicher, imaginarer Wurzeln wenn $D=0$, $a>0$, $a^2-4c=0$.

The equation $x^4 + \, x^2 + 16x + 9 \equiv (x+1)^2(x^2-2x-9)=0$ satisfies the second set of conditions above but does not have two pairs of equal roots nor are the roots all complex. The error is the same as that indicated in second note. Weber's first statement is correct, because for $a^2-4c=0$, D or $\triangle$ becomes, except for a constant factor, $b^2(27b^2-32a^3)$ and when a is negative this can vanish only when b is zero. Such is not the case however when a is positive.

NOTES ON THE THEORY OF NUMBERS.

By PROFESSOR L. E. DICKSON, The University of Chicago.

1. *Even perfect numbers.* Let $\sigma(k)$ denote the sum of all the divisors of k. A perfect number k is one for which $\sigma(k)=2k$. Let $k=2^n q$, where q is odd and $n>0$. Then $(2^{n+1}-1)\sigma(q)=2^{n+1}q$. Hence $\sigma(q)=q+d$, where $d=q/(2^{n+1}-1)$. Thus d is an integral divisor of q, so that q and d are the only divisors of q. Hence $d=1$ and q is a prime. Hence every even perfect number is of Euclid's type $2^n(2^{n+1}-1)$, where the second factor is a prime. This is much simpler than any proof that has been published hitherto.

2. *Amicable numbers.* In his elaborate memoir on amicable numbers, Euler* was led (§ 95) to the type $16pq$, $16.17.137r$, where p, q, r are distinct primes. These two numbers are amicable if and only if

$$p=m+9935,\quad q=n+9935,\quad r=4(m+n)+88799,\quad mn=2^7.3^4.7.23.73.$$

Since r always exceeds 100,000, the limit of the table of primes accessible to Euler, he made no discussion of this type. In view of the large number of pairs of factors of mn (only distinct even values of m, n need be examined), it seemed likely that there exist amicable numbers of this type. A complete examination of the 120 cases showed that p, q, r are all primes only when

$$m=2^4.3^3.7,\quad p=12959,\quad q=50231,\quad r=262079;$$
$$m=2^3.3.7,\quad p=10103,\quad q=735263,\quad r=2990783.$$

The cases in which numbers exceeding 10 million had to be considered are $m=6$, $r\equiv 0 \pmod 7$; $m=32$, $r\equiv 0 \pmod 5$. The two new pairs of amicable numbers are thus:

$$16.12959.50231,\quad 16.17.137.262079;\quad 16.10103.735263,\quad 16.17.137.2990783.$$

These were checked to be amicable and the primes rechecked by Lehmer's table.

3. *Bernouillian numbers* are most conveniently employed in the symbolic notation of Lucas; we have $(b+1)^n-b^n=0$. In this notation, the generalization by Stern (Munich Akad., 1877, p. 157) of Seidel's recursion formula may be written:

$$\left\{\binom{n}{1}+\binom{m}{1}\right\} b^{m+n-1} + \left\{\binom{n}{2}-\binom{m}{2}\right\} b^{m+n-2} + \left\{\binom{n}{3}\right.$$

* *Opuscula varii argum.*, 2, 1750, p. 23; *Commentationes Arithmeticae Collectae*, 1849, pp. 102-145.

$$+\binom{m}{3}\Big\} b^{m+n-3}+\ldots+\Big\{\binom{n}{n}+(-1)^{n-1}\binom{m}{n}\Big\} b^{m}=0,$$

for $n\geqq m$, where the binomial coefficient $\binom{m}{k}$ is zero if $k>m$. Summing separately the first terms and the second terms, we get

$$b^m(b+1)^n-b^n(b-1)^m=0.$$

To give a direct proof of the latter, we note that by definition

$$f(b+1)=f(b)+f'(0),$$

for any polynomial $f(x)$. Taking $f(x)=x^n(x-1)^m$, $n>1$, we obtain the desired formula.

Setting $\beta^m=2(2^m-1)b^m$, we may similarly write the other formula of Seidel and Stern as follows:

$$\beta^m(\beta+1)^n+\beta^n(\beta-1)^m=0, \quad n\geqq m>1.$$

4. $A^3+B^3+C^3=D^3$. Euler's general solution (*Comm. Arith.* 1, p. 199) is

$$A=n(gf-e^2), \quad B=n(hf+e^2), \quad C=n(f^2-ge), \quad D=n(f^2+he),$$

where we have employed the abbreviations

$$e=a^2+3b^2, \quad f=d^2+3c^2, \quad g=3ac+3bc-ad+3bd, \quad h=3ac-3bc+ad+3bd.$$

These abbreviations enable us to point out the identity which underlies his solution. In

$$A^3+B^3+C^3-D^3=n^3(f^3-e^3)[g^3+h^3-3ef(g+h)]$$
$$=n^3(f^3-e^3)(g+h)(g^2-gh+h^2-3ef),$$

it is the final factor which vanishes. This follows from the identity

$$ef\equiv(ad-3bc)^2+3(ac+bd)^2=\left(\frac{h-g}{2}\right)^2+\left(\frac{h+g}{6}\right)^2,$$

which in turn follows from $(a+b\sqrt{-3})(d+c\sqrt{-3})=ad-3bc+(ac+bd)\sqrt{-3}$.

We may also make use of the factorizations (where $\omega^3=1$):

$$A+C=(f-e)(g+f+e),\quad A+\omega C=(f-\omega e)(g+\omega f+\omega^2 e),$$
$$A+\omega^2 C=(f-\omega^2 e)(g+\omega^2 f+\omega e),$$
$$D-B=(f-e)(-h+f+e),\quad \omega D-B=(f-\omega e)(-h+\omega f+\omega^2 e),$$
$$\omega^2 D-B=(f-\omega^2 e)(-h+\omega^2 f+\omega e).$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

352. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve the equations, $x^3=-8y+24\dots(1)$.
$y^3=-8x+24\dots(2)$.

Solution by S. LEFSCHETZ, Clark University.

The curves represented by (1) and (2) have nine common points of which three are on the line $x-y=0$, since they are evidently symmetric with respect to it. For these points, then, $x^3+8x-24=0$.

One solution is $x=y=2$; dividing by $x-2$ we obtain, $x^2+2x+12=0$.
$\therefore x=y=-1+i\sqrt{11}$, and $x=y=-1-i\sqrt{11}$. To find the solutions, six in number, for which $x\neq y$, we multiply (1) by x, (2) by y, subtract and divide by $x-y$, so that we have,

$$(x+y)(x^2+y^2)=24\dots(3).$$

Also subtracting (2) from (1), and dividing by $(x-y)$, we have

$$x^2+y^2=8-xy\dots(4),$$

so that (3) can be written,

$$(x+y)(8-xy)=24\dots(5),$$

which can also be written,

$$xy(x+y)=8(x+y)-24,$$

and if we add to this last equation after multiplying it by 3, equations (1) and (2), we obtain,

$$x^3+y^3+3xy(x+y)=16(x+y)-24=(x+y)^3.$$

If then $x+y=t$, $xy=u$, we have, $t^3-16t+24=0$, $u=8-24/t$. The cubic can be written,

$$(t-2)(t^2+2t-12)=0.$$

For $t=2$, we have $u=8-\frac{24}{2}=-4$, so that the corresponding values of x or y are roots of $z^2-2z-4=0$; hence one system of values is $x=1+\sqrt{5}$, $y=1-\sqrt{5}$, and another, by permuting x and y.

The other factor of the equation in t has for roots,

$$t_1=-1+\sqrt{13},\quad t_2=-1-\sqrt{13},$$

to which corresponds,

$$u_1=6-2\sqrt{13},\quad u_2=6+2\sqrt{13}.$$

To (t_1, u_1) correspond the roots of $z^2-(-1+\sqrt{13})z+6-2\sqrt{13}=0$, or

$$x=\sqrt{13}-1+\sqrt{[-6\sqrt{(13)}-10]},\quad y=\sqrt{13}-1-\sqrt{[-6\sqrt{(13)}-10]},$$

and the point obtained by permuting x and y.

To have the values of (x, y) corresponding to (t_2, u_2) it is sufficient to change the signs affecting $\sqrt{13}$, in the two preceding ones, and we obtain

$$x=\tfrac{1}{2}[-\sqrt{(13)}-1+\sqrt{[6\sqrt{(13)}-10]},$$
$$y=\tfrac{1}{2}[-\sqrt{(13)}-1-\sqrt{[-6\sqrt{(13)}-10]},$$

and the point obtained by permuting x and y.

Also solved by Jeannette Brooks, Nellie Wood, J. Scheffer, V. M. Spunar, A. H. Holmes, and the Proposer.

353. Proposed by DANIEL KRETH, Oxford, Iowa.

Divide 2940 into two such factors that the square of one factor minus 21 will equal three times the other factor.

Solution by J. K. ELLWOOD, Kansas City, Mo.

Let x be one factor, then $2940/x$ is the other.

By the conditions of the problem, $x^2-21=3\times2940/x=8820/x$.

Whence, $x^3-21x=8820$. Multiplying both sides of the equation by x, $x^4-21x^2=8820x$. Adding $441x^2$ to both sides, $x^4+420x^2=441x^2+8820x$.

Completing squares, $x^4+420x^2+210^2=441x^2+8820x+210^2$, or $(x^2+210)^2=(21x+210)^2$.

Whence, $x^2=21x$, $x=0$, the introduced root, 21, and $\frac{1}{2}[-21\pm\sqrt{(-1239)}]$. Hence, 21 is the only possible real value for x. Therefore, $2940/21=140$, the other factor. Hence 21 and 140 are the factors.

Also solved by H. Prime, V. M. Spunar, S. Lefschetz, J. Scheffer, and S. G. Barton.

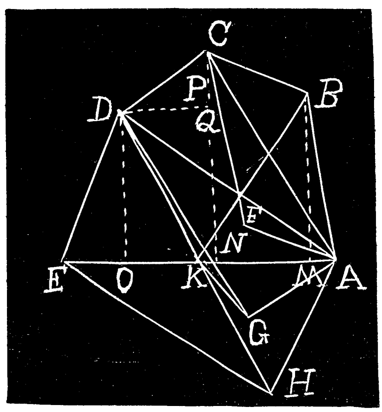
GEOMETRY.

377. Proposed by S. A. COREY, Hiteman, Iowa.

Let AB, BC, CD, DE, EA be the sides of a pentagon, plane or gauche. From A draw AF, AG, AH, parallel to, and of the same length and currency as BC, CD, DE, respectively. Bisect AE at K. Draw KB, KF, KG, and KH. Prove that $KB^2+KF^2+KG^2+KH^2=AB^2+BC^2+CD^2+DE^2$.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

K being the mid-point of the diagonal AE in the parallelogram $ADEH$, DKA must be a straight line, viz., the other diagonal. Let the orthogonal coordinates of B, F, G, H, with reference to AE as the axis of X, and A as origin, be, respectively, x_1, y_1; x_2, y_2; x_3, y_3; x_4, y_4; and $AK=EK=a$. Then



$AB^2+AF^2+AG^2+AH^2=(x_1^2+y_1^2)+(x_2^2+y_2^2)+(x_3^2+y_3^2)+(x_4^2+y_4^2)$..... (I),

and $KB^2+KF^2+KG^2+KH^2$

$$=[(x_1-a)^2+y_1^2]+[(x_2-a)^2+y_2^2]+[(x_3-a)^2+y_3^2]+[(x_4-a)^2+y_4^2]$$
$$=(x_1^2+y_1^2)+(x_2^2+y_2^2)+(x_3^2+y_3^2)+(x_4^2+y_4^2)$$
$$-2a[(x_1+x_2+x_3+x_4)-2a]\ldots\text{(II)}.$$

Letting fall the perpendiculars BM, CN, DO upon AE, and BP, DQ upon CN, we have $AB\cos BAM+BC\cos CBP+CD\cos CDA+DE\cos DEO=AB\cos BAM+AF\cos FAK+AG\cos KAG+AH\cos KAH=AE=2a$.

$\therefore x_1+x_2+x_3+x_4=2a$; therefore in (II), $x_1+x_2+x_3+x_4-2a=0$, and $AB^2+AF^2+AG^2+AH^2=KB^2+KF^2+KG^2+KH^2=AB^2+BC^2+CD^2+DE^2$. Q. E. D.

II. **Solution by the PROPOSER.**

Let a, b, c, and d be the vector sides AB, BC, CD, and DE, respectively. Then will $EA=-(a+b+c+d)$, $AK=\frac{1}{2}(a+b+c+d)$, $KB=\frac{1}{2}(a-b-c-d)$, $KF=\frac{1}{2}(-a+b-c-d)$, $KG=\frac{1}{2}(-a-b+c-d)$, and $KH=\frac{1}{2}(-a-b-c+d)$.

Squaring these vector expressions for KB, KF, KG, and KH, and adding, their sum is found to be $a^2+b^2+c^2+d^2$. As the square of a vector equals minus the square of its tensor, the truth of the proposition is demonstrated. Observe that a line drawn from D to K is equal to and may be substituted for KH in the equation of the problem.

378. **Proposed by G. I. HOPKINS, A. M., Instructor in Mathematics and Astronomy, Manchester High School Manchester, N. H.**

In the triangle AED, the lines BE and CE are drawn to the points B and C in the base of the triangle. If $AE=100$, $ED=125$, $BC=60$, and $\angle AEC=\angle BED=$a right angle, compute AB, BE, EC, and CD.

Solution by A. H. HOLMES, Brunswick, Maine.

Put $DAE=\theta$ and $ADE=\psi$. Let fall perpendicular EF on base AD. Then we have, $100\sin\theta=125\sin\psi$ or $4\sin\theta=5\sin\psi$... (1).

Since BED and AEC are right angles, $BD=\dfrac{DE}{\cos\psi}$, and $AC=\dfrac{AE}{\cos\theta}$.

$\therefore \dfrac{100}{\cos\theta}+\dfrac{125}{\cos\psi}-60=100\cos\theta+125\cos\psi$... (2).

Eliminating $\sin\theta$ and $\cos\psi$ from (1) and (2), and reducing,

$$\cos^6\psi+.96\cos^5\psi-3.1296\cos^4\psi-.0256\cos^3\psi+1.94745\cos^2\psi-.1152\cos\psi=0.$$

Solving by Horner's method, $\cos\psi=.8851+$. $\therefore \cos\theta=.8134+$.

Then since $AC=122.94+$, $AB=62.94+$.

Similarly, $CD=81.22+$. Also, $BE=65.71+$, and $CE=71.51+$.

Also solved by J. Scheffer.

382. **Proposed by PROF. R. C. ARCHIBALD, Brown Uniuersity, Providence, R. I.**

Between the side of a given rhombus and its adjacent side produced, to insert a straight line of a given length and directed to the opposite corner. "Euclidean constructions" are particularly desired.

Remark by V. M. SPUNAR, M. and E. E., Chicago, Illinois.

This is the famous Pappus problem: Rhombo dato et uno latere producto aptare sub angelo exteriori magnitudine datum rectam lineam, quae ad oppositum angulum pertingat.

Pappus, and a certain number of mathematicians, among them Newton, Huygens, and Gergonne, solved the problem algebraically and geometrically. (See E. Pruvost, *Geomètrie Analytique*, t. I, pp. 18-28.)

The problem in the present form, proposed by Prof. R. C. Archibald

in *l'Intermediaire de Mathematiciens* as Question 3667, suggested to Prof. P. Barbarin (Paris) a more general problem, viz., *Mener par un point donné dans un angle une secante de longueur donnés,* of which a complete solution has been published by himself in *l'Enseignement Mathematique* (XIII^e, 1, 1911, pp. 15-23).

The investigation is carried out analytically, and the following conclusions have been drawn. The generalized problem is solved algebraically by an equation of the third or fourth degree, and graphically by the intersection of a circle with an hyperbola. Special cases have been shown where the general equation can be lowered to second degree (where the Euclidean construction is possible). There is also a particular case, where the problem is reduced to that of the tri-section of an angle.

A solution of this problem as No. 364 is given by C. N. Schmall on page 140-141, Vol. XVII of the MONTHLY.

MECHANICS.

253. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

R_1 and R_2 are ranges on a horizontal plane of particles projected with given velocity from A on the plane to pass through B. Show that $a(R_1+R_2)-R_1R_2=\frac{a^4}{c^2}$, where $c=AB$ and a is the horizontal projection of AB.

I. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

Let h be the distance of B above the horizontal plane, v the velocity of projection, α and β the two angles of projection which make the particles pass through B. Let $2v^2/g=m$.

$$R_1=\frac{m}{2}\sin 2\alpha=\frac{m\tan\alpha}{1+\tan^2\alpha},\quad R_2=\frac{m\tan\beta}{1+\tan^2\beta}.$$

$$R_1+R_2=\frac{m(\tan\alpha+\tan\beta)(1+\tan\alpha\tan\beta)}{(1+\tan^2\alpha)(1+\tan^2\beta)},\quad R_1R_2=\frac{m^2\tan\alpha\tan\beta}{(1+\tan^2\alpha)(1+\tan^2\beta)}.$$

The equation of the trajectory is

$$y=x\tan\alpha-\frac{x^2}{m}\sec^2\alpha=x\tan\alpha-\frac{x^2}{m}(1+\tan^2\alpha).$$

Since B lies on this, we have $h=a\tan\alpha-\frac{a^2}{m}-\frac{a^2\tan^2\alpha}{m}$.

Solving as a quadratic in tan α we have

$$\tan\alpha=\frac{m+\sqrt{(m^2-4hm-4a^2)}}{2a}.$$

Similarly,

$$\tan\beta=\frac{m-\sqrt{(m^2-4hm-4a^2)}}{2a}.$$

Whence $\tan\alpha+\tan\beta=\frac{m}{a}$, $\tan\alpha\tan\beta=\frac{(hm+a^2)}{a^2}$,

$$\tan^2\alpha+\tan^2\beta=\frac{(m^2-2hm-2a^2)}{a^2},$$

$$(1+\tan^2\alpha)(1+\tan^2\beta)=\frac{m^2}{a^4}(a^2+h^2)=\frac{m^2c^2}{a^4}.$$

$$a(R_1+R_2)-R_1R_2=\frac{a^4}{m^2c^2}.m^2[1+\tan\alpha\tan\beta-\tan\alpha\tan\beta]=\frac{a^4}{c^2}.$$

II. Solution by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

Let α=the angle of projection of one particle, and β the angle of the other; R_1, the range of the former, and R_2, the range of the latter; and v, the velocity of the particles. The equations of the paths of the two particles are

$$y=x\tan\alpha-\frac{gx^2}{2v^2\cos^2\alpha}, \text{ and } y=x\tan\beta-\frac{gx^2}{2v^2\cos^2\beta}$$

and the equation of the inclined plane is $y=x\tan\gamma=x.\frac{\sqrt{(c^2-a^2)}}{a}$. Now it is easily shown that

$$R_1=\frac{v^2\sin 2\alpha}{g}\ldots(1), \text{ and } R_2=\frac{v^2\sin 2\beta}{g}\ldots(2).$$

Solving the equation of the plane with each of the equations of the paths of the particles, we easily find that

$$\cos^2\alpha=\frac{ga(R_1-a)}{2v^2\sqrt{(c^2-a^2)}}\ldots(3), \text{ and } \cos^2\beta=\frac{ga(R_2-a)}{2v^2\sqrt{(c^2-a^2)}}\ldots(4).$$

From (1), $\sin 2\alpha=\frac{gR_1}{v^2}$, or $\sin\alpha\cos\alpha=\frac{gR_1}{2v^2}\ldots(5)$, and from (2),

$$\sin\beta\cos\beta=\frac{gR_2}{2v^2}\ldots(6).$$

Squaring (5), and dividing by (3), we have

$$\sin^2\alpha=\frac{gR_1^2\sqrt{(c^2-a^2)}}{2av^2(R_1-a)}\ldots(7).$$

Squaring (6) and dividing the result by (4), we have,

$$\sin^2\beta=\frac{gR_2^2\sqrt{(c^2-a^2)}}{4av^2(R_2-a)}\ldots(8).$$

Equating the sum of (3) and (7), with the sum of (4) and (8), we have, after cancelling out common factors,

$$\frac{R_1^2\sqrt{(c^2-a^2)}}{2a(R_1-a)}+\frac{a(R_1-a)}{2\sqrt{(c^2-a^2)}}=\frac{R_2^2\sqrt{(c^2-a^2)}}{2a(R_1-a)}+\frac{a(R_2-a)}{2\sqrt{(c^2-a^2)}}.$$

Whence, $\dfrac{R_1^2(c^2-a^2)}{R_1-a}+a^2(R_1-a)=\dfrac{R_2^2(c^2-a^2)}{R_2-a}+a^2(R_2-a)$, or

$$\left[\frac{R_1^2}{R_1-a}-\frac{R_2^2}{R_2-a}\right](c^2-a^2)=a^2(R_2-R_1),$$

$$[R_1^2(R_2-a)-R_2^2(R_1-a)](c^2-a^2)=a^2(R_2-R_1)(R_2-a)(R_1-a),$$

$$[R_2R_2(R_1-R_2)-a(R_1^2-R_2^2)](c^2-a^2)=a^2(R_2-R_1)(R_2-a)(R_1-a).$$

$$\therefore[a(R_1+R_2)-R_1R_2](c^2-a^2)=a^2[R_1R_2-a(R_1+R_2)+a^2].$$

$$\therefore[a(R_1+R_2)-R_1R_2=a^4/c^2.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

Edited by Dr. G. E. Wahlin, University of Illinois.

178. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Find a formula which gives all the integral solutions prime to 5 of the congruence $x^2+y^2\equiv0\pmod{5^4}$.

II. **Solution by E. B. ESCOTT, Ann Arbor, Michigan.**

The solution of the congruence $x^2+y^2\equiv 0 (\text{mod } 5)$, is $x\equiv \pm 2y (\text{mod } 5)$. Let $x=5a\pm 2y$, and substitute in $x^2+y^2\equiv 0$ (mod 5^2).

We get $x\equiv \pm 7y$ (mod 5^2), and proceeding in this way we get finally,

$$x\equiv \pm 182y \ (\text{mod } 5^4).$$

NOTE. A similar and more complete solution by the Proposer has already been published. ED. W.

180. **Proposed by A. H. HOLMES, Brunswick, Maine.**

Find integral values of x and y such that $96x-96y+21=\square$.

II. **Solution by B. KRAMER, E. M., Pittsburg, Pennsylvania.**

Let $x-y=z$. Then $96z+21=u^2$. From this it is easily seen that u is odd, and $u^2\equiv 5$ (mod 8).

But if we put $u=2k+1$, $u^2=4k(k+1)+1$, which, since k or $k+1$ is even, says that $u^2\equiv 1$ (mod 8). Hence no square can be congruent to 5 modulo 8, and the relation is impossible.

181. **Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.**

If $2n+1$ is an odd prime p, $(2n)!\equiv (-1)^n 2^{4n}(n!)^2$ (mod p^2).

Solution by S. LEFSCHETZ, Clark University.

The given congruence can be written

$$2n!\equiv (-1)^n 2^{2n}[2.4\ldots 2n]^2 (\text{mod } p^2), \text{ or}$$
$$1.3.5\ldots(2n-1)\equiv (-1)^n 2^{2n}.2.4.6\ldots 2n (\text{mod } p^2)$$
$$\equiv (-1)^n 2^{2n}(p-1)(p-3)\ldots[p-(2n-1)]$$
$$\equiv 2^{2n}[1.3\ldots(2n-1)-p.1.2\ldots(2n-1)(\tfrac{1}{1}+\tfrac{1}{3}+\ldots+\frac{1}{2n-1})] (\text{mod } p^2).$$

If we adopt Bachmann's notation [*Niedere Zahlentheorie*, Bd. 1, p. 161] and call $1/m$ the number m' such that $mm'\equiv 1$ (mod p). We have to prove, as can be seen at once that:

$$\frac{2^{2n}-1}{p}\equiv \tfrac{1}{1}+\tfrac{1}{3}+\ldots+\frac{1}{2n-1} (\text{mod } p),$$

and this is proved by Bachmann (loc. cit., p. 164).

182. **Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.**

Find two general solutions in integers of the equation $x^2=616318177y-1$.

No solution of this problem has been received.

183. **Proposed by M. T. GOODRICH, Dixfield, Maine.**

Show what relation must exist between the quantities A, B, and C, in the harmonic ratio $\dfrac{AB}{(A+B+C)(-C)}=-1$, so that they will be real positive integers.

Solution by S. LEFSCHETZ, Clark University.

This can be written $AB=(A+B+C)C$. Let δ be the greatest common divisor of A, B, C, and let $A=\delta A'$, $B=\delta B'$, and $C=\delta C'$. Substituting we have $A'B'=C'(A'+B'+C')$.

Let now $C'=p.q$, p being prime to B', and q to A'. Since p divides $A'B'$ and is prime to B', it divides A'. Let $A'=\lambda p$, and similarly $B'=\mu q$. Since A', B', pq have no common divisors, q must be prime to λ and p to μ. We have by substitution, $\lambda\mu=\lambda p+\mu q+pq$, or $2pq=(\lambda-q)(\mu-p)$, and p being prime to $\mu-p$, while it divides the right member, must necessarily divide $\lambda-q$. Let $\lambda-q=hp$, similarly, $\mu-p=kq$.

$\therefore hk=2$. Hence, either $h=1$ and $k=2$, or $k=1$ and $h=2$.

On account of the symmetry in notations it is sufficient to consider the case $h=1$, $k=2$. Then $\lambda=p+q$, $\mu=p+2q$.

$\therefore A=\delta p(p+q)$, $B=\delta q(p+2q)$, $C=\delta pq$, is the general solution, p and q being prime to each other. This can be verified easily by substitution.

Also solved by the Proposer.

AVERAGE AND PROBABILITY.

202. **Proposed by F. P. MATZ, Ph. D., Reading, Pa.**

If three chords are drawn at random in a circle, what is the chance the center of the circle is enclosed by the three chords, and what is the mean area of this enclosing triangle?

Solution by the late G. B. M. ZERR, Ph. D.

Let AB, CD, EF be the chords intersecting in Q, P, and R, respectively. Let O be the center. Draw the perpendiculars OG, OH, OL to AB, CD, EF, respectively. Also draw OA, OB, OC, OD, OE, OF, OQ, OP, OR.

Let $\angle AOG=\theta$, $\angle COH=\phi$, $\angle EOL=\psi$, $\angle HOG=\rho$, $\angle HOL=\mu$, $\angle LOG=\omega$, $\angle POQ=\delta$, $\angle QOR=\gamma$, $OP=x$, $OQ=y$, $OR=z$, a=radius of circle. The limits of $\theta=0$ and $\frac{1}{2}\pi$; of ϕ, 0 and θ; of ψ, 0 and ϕ; of ρ, $2\pi-2\psi-\theta-\phi=\rho'$ and $\pi-\theta-\phi=\rho''$; of μ, $2\pi-2\theta-\phi-\psi=\mu'$ and $\pi-\phi-\psi=\mu''$; of ω, $2\pi-2\phi-\theta-\psi=\omega'$ and $\pi-\theta-\psi=\omega''$; of δ, 0 and π; of γ, 0 and π; of x, 0 and a; of y, 0 and a; of z, 0 and a. Let A=average area, C=the required chance.

$$\therefore C=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\phi}\int_{\rho''}^{\rho'}\int_{\mu''}^{\mu'}\int_{\omega''}^{\omega'} d\theta\, d\phi\, d\psi\, d\rho\, d\mu\, d\omega}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\phi}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} d\theta\, d\phi\, d\psi\, d\rho\, d\mu\, d\omega}$$

$$=\frac{48}{\pi^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\phi}[(\pi-2\theta)(\pi-2\phi)(\pi-2\psi)]d\theta\, d\phi\, d\psi=\tfrac{1}{8}.$$

$$A=\frac{\frac{1}{2}\int_0^a\int_0^a\int_0^a\int_0^\pi\int_0^\pi[xy\sin\delta+yz\sin\gamma-xz\sin(\delta+\gamma)]xdx\, ydy\, zdz\, d\delta\, d\gamma}{\int_0^a\int_0^a\int_0^a\int_0^\pi\int_0^\pi xdx\, ydy\, zdz\, d\delta\, d\gamma}$$

$$=\frac{8}{\pi a^6}\int_0^a\int_0^a\int_0^a(x+z)xy^2z\, dx\, dy\, dz=\frac{8a^2}{9\pi}.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

357. **Proposed by V. M. SPUNAR, M. and E. E., Chicago, Illinois.**

Solve the system

$$\sqrt{(x^2+a^2+b^2+c^2)}=\sqrt{(y^2+b^2+c^2)}+\sqrt{(z^2+b^2+c^2)},$$
$$\sqrt{(y^2+a^2+b^2+c^2)}=\sqrt{(x^2+c^2+a^2)}+\sqrt{(z^2+c^2+a^2)},$$
$$\sqrt{(z^2+a^2+b^2+c^2)}=\sqrt{(x^2+a^2+b^2)}+\sqrt{(y^2+a^2+b^2)}.$$

358. **Proposed by V. M. SPUNAR, M. and E. E., Chicago, Illinois.**

Show that $\dfrac{n(n+1)\ldots(n+m-1)}{m!}-n\dfrac{n(n+1)\ldots(n+m-4)}{(m-3)!}+\dfrac{n(n-1)}{2!}\cdot$
$\dfrac{n(n+1)\ldots(n+m-7)}{(m-6)!}-\ldots=0$, if $m>2n$; and $=1$ if $m=2n$.

359. **Proposed by V. M. SPUNAR, M. and E. E., Chicago, Illinois.**

Show when $1/(1-x)(1-x^3)(1-x^5)(1-x^7)\ldots=(1+x)(1+x^2)(1+x^3)$ $(1+x^4)\ldots$

GEOMETRY.

390. **Proposed by PROF. R. C. ARCHIBALD, Brown University, Providence, R. I.**

Find, geometrically and without introducing focal properties, the locus of the vertices of the conjugate parallelograms of an ellipse.

391. **Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.**

An ellipse is inscribed in the triangle of reference and has one focus at (sec$A$, sec$B$, sec$C$). Find the other focus and the sum of the squares of the axes of the ellipse.

392. **Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.**

A tangent to a curve at any point P cuts the tangent and the normal at a fixed point O in the points M and N, and the rectangle $OMP'N$ is completed. Find the curve which is such that the triangle formed by the tangents at any three points P, Q, R is equal to the triangle formed by the corresponding points P', Q', R'.

CALCULUS.

313. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Evaluate the definite integral $\int_0^\infty (e^{-2ax^2}+e^{2ax^2})\,dx$.

314. Proposed by REV. J. H. MEYER, S. J., New Orleans, La.

A fox started from a certain point and ran due east 300 yards, when it was overtaken by a hound that started from a point 100 yards due north of the fox's starting point, and ran directly towards the fox throughout the race. Find the lengh of the curve described by hound, both having started at the same instant, with a uniform velocity.

315. Proposed by C. N. SCHMALL, New York City.

If $y=f(x)$, show by Taylor's Theorem that

$$f\left(\frac{x}{1+x}\right)=y-\frac{x^2}{1+x}\cdot\frac{dy}{dx}+\frac{x^4}{2\,(1+x)^2}\cdot\frac{d^2y}{dx^2}-\frac{x^6}{2.3.\,(1+x)^3}\cdot\frac{d^3y}{dx^3}+\dots\ \text{etc.}$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

184. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Prove that $\frac{\pi}{12}=\tan^{-1}\frac{1}{2^2}+\tan^{-1}\frac{1}{8^2}+\tan^{-1}\frac{1}{30^2}+\tan^{-1}\frac{1}{112^2}+\dots$, where 2, 8, 30, 112..., is a recurring series with the recursion formula $u_n=4u_{n-1}-u_{n-2}$.

NOTES AND NEWS.

Dr. R. D. Carmichael has been appointed to an assistant professorship in mathematics at the University of Indiana. S.

The one hundred and fifty-third regular meeting of the American Mathematical Society was held at the University of Chicago, on Friday and Saturday, April 28 and 29, 1911. There were in attendance eighty-six members and thirty visitors. Among the members from a distance were Professors Maxime Bôcher and William F. Osgood of Harvard University; Professor Percy F. Smith of Yale University; Professor F. N. Cole, Secretary of the Society, and Dr. N. J. Lennes of Columbia University; and Professor H. B. Fine, President of the Society, of Princeton University. The large attendance was due in part to the fact that this was the first meeting of the Society as a whole, aside from the summer meetings, held in the West since the sessions in St. Louis, at the time of the World's Exposition; but more largely, doubtless, to the desire of the members to hear the retiring presidential address of Professor Bôcher, who was greated by an audience of over one hundred members and visitors. At the dinner on Friday evening, seventy-six members were present and remained till a late hour in social intercourse. There were fifty-three papers presented at the four sessions. S.

At the Summer Session of the Ohio State University, Columbus, Ohio, June 19 to August 11, there will be courses in elementary algebra, plane and solid geometry, plane trigonometry, analytic geometry, differential and integral calculus, and in the teaching and history of mathematics. S.

At the University of Illinois the following courses in mathematics will be given during the next Summer Session, the number of hours being *credit hours* in each case: Advanced algebra, three hours; plane trigonometry, two hours; analytic geometry, five hours; differential calculus, five hours; integral calculus, five hours; synoptic course, daily; theory of equations, daily. S.

At the University of Wisconsin the following courses in mathematics will be offered during the next Summer Session, the hours referring to the number of recitations per week: Elementary algebra, five hours; solid geometry, five hours; plane trigonometry, five hours; analytic geometry, five hours; differential calculus, five hours; complex numbers, three hours; elementary analysis, twelve hours; integral calculus, twelve hours; differential equations, five hours; projective geometry, three hours; differential geometry, three hours; calculus of variations, five hours; theory of integrals, five hours. S.

At the University of Michigan, the following courses in mathematics will be offered during the next Summer Session, the hours referring to the number of recitations per week: Elementary algebra, plane and solid geometry, each four hours; trigonometry, college algebra, and analytic geometry, each four hours; differential and integral calculus, each four hours; introduction to mathematical theory of finance, insurance, and statistics, four hours; courses for teachers in geometry and algebra, and history of mathematics; mechanics, differential equations, projective geometry, and theory of functions of a real variable, each four hours. S.

Miss Mary C. Spencer, of Sophie Newcomb College for Women at New Orleans, Louisiana, has been influential in organizing a Mathematical Club among the secondary school teachers of the city. The club holds regular semi-monthly meetings for its members, and once a year holds an open meeting for the public school teachers of New Orleans. Such a gathering of several hundred teachers was recently addressed by Professor H. E. Slaught of the University of Chicago. F.

The Summer Quarter at the University of Chicago will extend from June 19 to September 1, being divided into two terms, the second beginning July 27. The following courses in mathematics will be offered: Trigonometry, college algebra, plane analytic geometry, solid analytic geometry, differential calculus, integral calculus, elliptic integrals, finite collineation groups, calculus of variations, integral equations, general analysis, synthetic projective geometry, analytic mechanics, vector analysis, introduction to celestial mechanics, reading and research in pure and applied mathematics, teaching of elementary school mathematics, teaching of secondary-school mathematics, and critical review of secondary mathematics. All elementary courses are given five hours per week and advanced courses four hours per week. All courses count for regular college or graduate credit. S.

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The author of the following article, Mr. Yoshio Mikami, has already acquired an enviable reputation as a student in the history of mathematics of Japan. Last year he published in the *Abhandlungen zur Geschichte der Mathematischen Wissenschaften* a work entitled "Mathematical Papers from the East." In this work he has included upwards of fifty articles showing the present interest of mathematicians in Japan, not merely in the history of mathematics, but in analysis, theory of numbers, modern geometry, theory of functions, and other topics of this nature. He is a student and friend of the learned historian of mathematics, Mr. Endo. He has not connected himself with any of the faculties of the Japanese universities, preferring to live the life of an independant savant. He has contributed extensively to the *Bibliotheca Mathematica* and the *Nieuw Archief voor Wiskunde* and ranks with Professor Hayashi as one of the best known of the younger historians of mathematics in the East. Having myself become much interested in the history of Oriental mathematics, and having made a large collection of early printed works and manuscripts on this subject, it was a pleasure to me to have Doctor Paul Carus suggest that I collaborate with Mr. Mikami in a history of Japanese mathematics. This I have done, and it is expected that the work will appear during the coming year. Mr. Mikami has made some valuable translations that set forth the nature of the native mathematics in his country. He is an indefatigable worker, and the West is already much in his debt and will be much more so in the next few years.

DAVID EUGENE SMITH.

THE TEACHING OF MATHEMATICS IN JAPAN.

By YOSHIO MIKAMI, Okara in Kazusa, Japan.

For the last two or three centuries there flourished a peculiar style of mathematics in Japan, which we distinguish from the European mathematics by the name *Wasan* or Japanese mathematics. It was first learned from China, yet further developments were effected quite independent of the Chinese.* But Japan saw a political change in 1868, which is well known

* As to the history of Japanese mathematics, a work will shortly appear under the joint authorship of Dr. David Eugene Smith and Yoshio Mikami.

as the *Ishin* or Restoration. The country was opened to commerce with Western powers and the Occidental civilization then poured in. Old schools were closed and many new ones were established, where mathematics were taught in the Western style, Japanese mathematics of former days being neglected. Those who had been schooled in the old style were obliged to learn the new science, and the state of things has changed in a short time to assume an appearance in accord with the rest of the civilized world. At present only Occidental mathematics is taught in the schools. The only exception being the teaching of the *soroban* arithmetic in common schools. The *soroban* is an abacus that was brought from China some centuries ago and that has been used universally during the ages of the old Japanese mathematicians.

The schools established in Japan since the Restoration have often gone through reorganizations, but this is not the place to give details about these changes. Suffice it to say, the main system of education has always remained untouched in spite of the frequent changes, being composed of the common school, the middle school, and the university. At present there is the higher school between the last two.

The common school provides ordinary and higher courses, the former extending through six years and the latter through two or three additional years. When boys and girls reach the age of six they are required to attend the ordinary common school and to remain there for six years. The arithmetic taught in this school consists of the following subjects:

First year: Numeration of numbers up to 100. Mental addition, subtraction, rudimentary multiplication and division.

Second year: Mental calculations with numbers up to 1000, including the four operations.

Third year: Numeration of numbers up to 10,000. The four operations in writing.

Fourth year: Four operations on numbers up to a hundred million. Calculations with various measures. Decimal fractions.

Fifth year: Four operations on whole numbers and decimals. Areas and volumes. Arithmetic with various measures; the metric system, foreign measures.

Sixth year: Four operations with fractions. Problems in ratio and rates. The *soroban* arithmetic may be taught in this year, but it may also be dispensed with. In the majority of schools, however, it is actually being taught.

It must be admitted that the Japanese system of numeration is much simpler than in the newer form expressed in the European languages, and this is a reason why the young Japanese are taught more arithmetic than is the case in Europe and America. The class-books of common schools are all prepared by the State under the direction of the Department of Education. The chief editor for arithmetical treatises is S. Iijima.

The compulsory education ends with the ordinary common schools, and the graduates of these schools are admitted to middle schools, though not infrequently an examination is required on account of the large number of applicants. The higher common school is intended for those who have to complete their education here. Its course regularly consists of two years, but it may extend to three years. The arithmetic in this school is roughly as follows:

First year: Greatest common measure, least common multiple, calculations with fractions in a more general sense than in the ordinary course, problems of application, problems in rates, simple and inverse proportions, and percentage partition.

Second year: Proportion, including compound proportion; drill in review of previous work. Geometrical drawing is also taught.

In the *third year*, when given, some further lessons in arithmetic are provided.

The middle schools extend through five years, the graduates of ordinary common schools being admitted. The lessons in mathematics are distributed somewhat as in the following table:

	1st Year	2nd Year	3rd Year	4th Year	5th Year
Arithmetic	3 3 3	3 3			
Algebra		3	2 2 2	2 2 2	2
Geometry			3 3 3	3 3 3	2 2 2
Trigonometry					2 2

The figures indicate the number of hours per week and the dotted lines indicate the divisions of terms, one school year consisting of three terms.

It is a matter of course that the distribution of lessons varies according to the different schools. The above is only a specimen.

In arithmetic, the following subjects are taught: Calculation of whole numbers, properties of the same, calculation of fractions, decimals, measures, money, etc. Simple, inverse, and compound proportions, problems in rates, percentage, profit and loss, interest.

The algebra course is somewhat similar to that given in the briefer of

Charles Smith's two treatises; including the four operations, indices, linear operations in one unknown, linear simultaneous equations, fractions, quadratic equations, irrational equations, extraction of square and cube roots, proportion, arithmetic and geometric progressions, interest, etc.

The geometry course is modeled after the treatise published by the Association for the Improvement of Geometrical Teaching in England. Proportion is treated according to the algebraic method. Solid geometry follows closely the outline of geometry in Wilson's treatise.

The trigonometry course follows closely the contents of John Casey's little work.

The text-books of the middle school are not prepared by the State as in the case of the common school, but are left to private authors. Nevertheless they are examined by the Department of Education, before they are authorized. There is a considerable number of these text-books now in use in Japan. Among those the most popuular are D. Kikuchi's Geometry, R. Fujisawa's Arithmetic, the same author's Algebra, etc. We are greatly indebted to the authors of these able treatises for the success of mathematics teaching in our country.

In the Japanese middle school all the boys are taught in the same manner, even though they may differ widely in their aims. Those who have to complete their education in the school and those who are preparing for higher grades of education are all instructed in the some school and all in the srme manner. Even the literary and scientific courses are not distinguished. Consequently the number of subjects taught becomes exceedingly large. And, moreover, the interrelations between these numerous subjects receive little attention. Even the different branches of mathematics are taught quite independently and irrespective of other branches. We hope however that such a defect as this will be remedied in the future, as a new program for the work of the schools is being prepared by the Department of Education, which will be made public very soon.

Before the graduates of the middle school may enter the university they must go through another school, whose course lasts three years. This school and other higher professional schools are all open to the graduates of middle schools, but as there is an insufficient number of these schools the candidates are obliged to pass competitive examinations, which proves a great barrier for young men. On this account they are naturally induced to pay most attention to the results of their examinations and the standing of these schools is determined solely by this criterion. It results therefore that the lessons are practically regulated with the sole aim of preparing for these examinations, which have thus acquired such an importance that they directly influence the teaching of the middle schools. It is consequently necessary to say something about these examinotions. As specimens we quote the problems given last summer in the entrance examination for the higher schools.

Algebra. 1. If m arithmetic means be inserted between two numbers a and b, it is required to find the rth term counting from a on.

2. A certain man leased land for 144 yen, and keeping 18 units of it for his own cultivation, he sublet the rest to a third man at a gain of 0.20 yen per unit of land. Then the amount he receives just paid his own rent. Required the number of units of land leased by him.

3. Given the relations $\frac{bz+cy}{b-c}=\frac{cx+az}{c-a}=\frac{cy+bx}{a-b}$, prove that $(a+b+c)(x+y+z)=ax+by+cz$.

4. Determine the values of p, q, r such that the coefficients of x^5, x^3, and x shall vanish in the expansion of $(x^3-px^2+qx-r)(px^3+x^2+5x+7)$.

Geometry. 1. If from a point P outside a circle whose center is C two tangents PA and PB are drawn to the circle, and if any chord MN is drawn through the middle point of the chord AB, it is required to prove that the four points P, C, M, N are concyclic and that the line joining the point P and the center C bisects the angle MPN.

2. Prove that the area of a quadrilateral is constant, if the lengths of its two diagonals and the angle included by them are constant.

3. There is a solid angle with three faces, whose diedral angles are each right angles. If the solid angle be cut by a plane, the orthocenter of the triangle formed by the intersections of the cutting plane and the faces of the solid angle is the same point as the foot of the normal dropped from the vertex of the solid angle to the plane.

Trigonometry. This examination included also questlons in trigonometry which are not here given.

Since the examinations are held for the purpose of making selections from among the graduates of middle schools, they ought to be in accordance with the subjects taught in these schools. But we are told that sometimes little attention is paid to this. As a rule, emphasis is laid on arithmetic, though nothing in this line has appeared in the above problems for entrance to higher schools. As the subject is taught only in the first years of the middle school, those who go to the examination rarely remember correctly what they have learned so long before. Therefore there are schools where arithmetic is repeated in the highest class or the fifth year, which proves a useless waste of time and labor for the students. For what need is there of using the purely arithmetical knowledge for those who are able to treat the same more simply by the aid of algebra?

Some time ago R. Fujisawa powerfully and ably maintained that arithmetic is the most difficult subject to teach among the various branches of mathematics, and consequently that most attention should be paid to it. This reasonable claim was unfortunately misunderstood as if meant to lay more stress on the subject of arithmetic than on algebra. The result was unfortunate, and we hope the abuse will soon be removed.

In the common school boys and girls are taught alike, but boys only are admitted to the middle school. Corresponding to the latter, however, there are girls' high schools, whose courses are not unique, consisting of three, four, or five years according to circumstances. The girls mostly finish their education in these schools, which are open to the graduates of the ordinary common school. The branches of mathematics taught in the girls' high school are, arithmetic, algebra, and geometry,—all the lessons being simpler than in the middle school. Sometimes algebra or geomety is withdrawn, and solid geometry is never given.

In a certain one of the girls' high schools mathematics is taught two hours per week to all the classes, as follows:

First year: Whole numbers and decimals, various measures, fractions.

Second year: Whole numbers, fractions and decimals, ratio and proportion, percentage.

Third year: Ratio and proportion, percentage, extraction of square root.

Fourth year: Elementary algebra.

Fifth year: Elementary geometry.

The normal school is a school of almost the same grade as the middle school and the girls' high school, and it is here that the teachers of common schools are trained. Most of the normal schools are so constituted that both boys and girls are admitted, although they are taught in different classes; but there are some schools established for boys or girls only. A program for the lessons in mathematics of the normal school was issued recently by the Department of Education. In it we find the interrelation of the various branches of the science emphasized, and the whole has been made more practical, certainly no small advancement in the teaching of mathematics. We hope that the program for the middle school will be arranged with the same aim.

The Higher Normal Schools, which are located in Tokyo and Hiroshima, are schools where the male teachers of the middle class schools are taught. The graduates of the normal school or of the middle school are admitted, and the preliminary course consists of one year, and the main course of three years. The preliminary course is the same for all students of the school, and the mathematics consists of arithmetic, algebra, and geometry, each two hours per week. These lessons are reviews of what has been given in the middle school or in the normal school.

In the main course there is a department for mathematics, physics, and chemistry. The teachers of mathematics are trained here. The department is subdivided into two different classes,—in the first of which mathematics and physics are mainly taught, while in the other physics and chemistry are emphasized.

The subjects of mathematics taught in the first class are as follows:

First year: Algebra, two hours, including series, continued fractions and determinants. These are treated a little more concisely than in C. Smith's *Treatise on Algebra.*

Geometry, two hours, comprising the advanced part of elementary geometry, together with modern geometry. Also trigonometry, two hours.

Second year: Theory of equations, two hours; analytic geometry, four hours, together with seminary exercises.

Third year: The calculus, six hours; solid analytic geometry and dynamics.

In the second class mathematics is taught in a more practical way, its lessons ending in the second year.

F. Sembon and T. Hayashi are the chief professors of the Tokyo Higher Normal School. T. Takehashi teaches in the Hiroshima School.

There are two other schools of the same nature for young women, one in Tokyo, and the other at Nara. In the Tokyo school the students are taught in three separate departments, — letters, science, and arts, each lasting for four years. Mathematics is taught in the science department, and the subjects are arithmetic, algebra, geometry with geometrical drawing, trigonometry, and the teaching of mathematics. These lessons are distributed among the four years, four hours per week in the first year, and three hours in other years.

In the Nara school the science department is subdivided into two courses, in one of which mathematics, physics, and chemistry are taught for the main subjects, and in the other mainly history and mathematics. The same subdivision is to be adopted in the Tokyo school in the coming academic year. The professor of mathematics in the Tokyo Female Higher Normal School is L. Mori.

Besides the main courses in these normal schools sometimes special courses are opened, where the training will end in two or three years.

Further, there is at Sendai a Temporary School for Training Mathematical Teachers. The course extends through two years. K. Hakii is the professor.

Teachers of middle class schools are also granted certificates by special examinations held by the Department of Education. Graduates of middle class schools only are permitted to apply for them. They are divided into two parts, preliminary and final. The former is held by the local governments and the latter by the Department itself. Only those who have been successful in the preliminary examination can take the latter. In mathematics the knowledge of arithmetic, algebra, and geometry is tested in the preliminary examination, and the first part of the final examination is also devoted to the same subjects. Only those who pass the earlier examinations are admitted successively to the examinations in trigonometry, analytic geometry, and the calculus. A terchers' certificate is granted to those who have succeeded in arithmetic, algebra, and geometry. These examinations are the same for men and women, except that the candidates for teachers of female schools are permitted to answer a smaller number of the questions. A few only go up to the calculus.

Those who take these examinations come from various quarters, but for mathematics the Physics School in Tokyo is the main source of supply. This is a private school. The classes extend through three years or six semesters. In the first two years all the students are taught together, while in the last year, the courses are subdivided, the first in mathematics and the other in physical sciences. Mathematics is taught there in the first two years from arithmetic and geometry up to the calculus, and in the third year some more advanced topics of the same subjects are taught. To speak more definitely, these subjects are as follows:

First year: Arithmetic, including theory of prime numbers, etc.;

elementary algebra, including quadratic trinomials, determinants, etc.; elementary geometry, elementary trigonometry.

Second year: Elements of analytic geometry, plane and solid; spherical trigonometry; differential and integral calculus, including algebraic analysis and differential equations.

Third year: Arithmetic and algebra, or theory of numbers and theory of equations, etc.; modern geometry, analytical trigonometry, differential and integral calculus.

Besides these, physics, chemistry, astronomy, and surveying are taught.

The teachers in mathematics are S. Hayashi, Kanazawa, Kariya, Kuniyeda, Mimori, Noguchi, Ogura, Sakuma, Sakurai, Sawayama, Sembon, Terao, and Yasuda.

In the teachers' college of the Waseda University, Tokyo, which is a private school, mathematics is taught in the mathematics department and also in the physics and chemisty departments. In the first of these departments the subjects taught are as follows: Arithmetic, algebra, geometry, trigonometry, which are distributed throughout the whole of three years. Plane analytic geometry in the first year, and solid geometry in the second year. Differential calculus in the second year, and integral calculus including differential equations in the third year.

In the second of the departments, arithmetic, algebra, geometry, trigonometry, and analytic geometry are taught in the first year; solid analytic geometry, and the differential calculus in the second year; and integral calculus in the third year.

The higher schools, eight in number, are preparatory to the Imperial Universities, and are subdivided into three departments. Of these the second department is preparatory to the colleges of science, engineering and agriculture. There mathematics is taught as follows:

First year: Algebra, three hours; and plane trigonometry, two hours.

Second year: Conic sections, three hours; and theory of equations, one hour.

Third year: Calculus, four hours; and dynamics, two hours.

In these subjects the following works are used as text-books or rather as books of reference: C. Smith's *Treatise on Algebra,* subsequent to the chapter on combinations; Todhunter's *Plane Trigonometry;* Puckle's *Conic Sections;* Aldis's *Solid Geometry* (about one-fourth of the book); Burnside and Panton's *Theory of Equations* (the first half); Williamson's treatises on differential and integral calculus, or those of Todhunter; McGregor's *Dynamics.*

Mathematics is not taught to the agricultural students in the third year.

The higher schools are to be reorganized in the near future by the Department of Education although the plan proposed is contrary to pub-

lic opinion. According to the plan the schools will be made institutions where higher general education is given independently of the preparatory work for entrance to the universities, although the graduates are to be received in the latter as before. In that case the courses of instruction will be largely altered, the subject of mathematics being not excepted. The consequence will certainly be a great change in the teaching of the middle school.

The preparatory course of the private Keio University, Tokyo, corresponds to the first department of the higher school, but here mathematics is largely taught with the aim of training the students in the habit of logical thought. This is not generally done elsewhere. The course extends through two years, in the first of which analytic geometry is taught, and in the second the calculus. The chief teacher is S. Fukugawa.

There are two Imperial Universities in Japan, one in Tokyo, and the other in Kyoto. In the mathematics department of the College of Science of Tokyo the following subjects are taught:

First year: Calculus, five hours; analytic geometry, two hours; miscellany in elementary mathematics, two hours; astronomy and method of least squares, three hours; theoretical physics, mathematical seminary, three hours.

Second year: Theory of functions and elliptic functions, three hours; algebraic curves, three hours; differential equations, two hours; theory of numbers and higher algebra, four hours; dynamics, three hours; mathematical seminary and physical experiments.

Third year: Theory of functions and elliptic functions, three hours; higher geometry, two hours; algebra, three hours; differential equations, two hours; miscellany in higher analysis, two hours; dynamics, calculus of variation, three hours; mathematical lectures (optional).

In the departments of astronomy, theoretical physics and experimental physics, mathematics is also taught. In the first year of these departments the following subjects are taught: Calculus, geometry, mathematical exercises, method of least squares. In the second year mechanics, differential equations, etc. In the third year of the first two departments lessons in the theory of functions are given.

In the department of chemistry mathematics is taught for three hours in the first year.

The professors of mathematics are R. Fujisawa, T. Takagi, E. Sakai, and T. Yoshiye, and assistant professor S. Nakagawa.

In some classes of the Engineering College mathematics is being taught somewhat after the practical fashion of John Perry, formerly of the College.

In the Science and Engineering College of Kyoto there is a department of mathematics, wherein are taught the following subjects: Plane and solid analytic goemetry, higher geometry, calculus including differential

equations, theory of functions, higher algebra, theory of numbers, mechanics, theoretical physics, astronomy, physical experiments, and special lectures.

These subjects are taken by the students at convenient times. They are not prescribed for the given years as in the Tokyo University, certainly presenting some advantages over the latter.

The mathematical courses in the departments of engineering are: Integral calculus, three hours for four months; differential equations, two hours for six months; dynamics, two hours for a year; theory of errors, two hours for four months.

The instructors are professors K. Miwa, J. Kawai, assistant professors S. Yoshikawa and T. Wada, and lecturer T. Nishiuchi.

The College of Science of the North-Eastern Imperial University will be opened at Sendai in 1911, with T. Hayashi and M. Fujiwara for professors of mathematics.

In the privately founded universities there is established no colleges of science except in Waseda, the teachers' course of which we have already mentioned. The Tokyo University for Women contains a department for science, which may be mentioned as corresponding to the same in the Normal School for Women, though there will be found some differences in the subjects taught.

The mathematical teaching of some professional schools other than the universities, may also be of some interest.

Technological schools are established in Tokyo, Nagoya, Osaka, Kyoto, and Kumamoto, where the graduates of middle schools are admitted. The mathematics taught in these schools consists of: Algebra, supplementing the knowledge learned in the middle school and including combinations and permutations, series, logarithms, and differential coefficients; solid geometry and important curves; conic sections and solid analytic geometry; the rudiments of the calculus, sufficient to give familiarity with the symbols.

These subjects are taught in some departments only, and in others they are omitted wholly or partly. The head master of mathematics in the Tokyo school is M. Mimori.

The Higher Commercial School of Tokyo provides a one-year preparatory course, and a three-year main course. In the former, mathematics is taught three hours per week. The subjects are: Measures, proportion and interest, problems in equations, series, logarithms and their application, probability, etc. Commercial arithmetic is taught in the first two years of the main course, the professor of mathematics being G. Sawata. There are several other schools of the kind in the country.

Mathematics is not taught in the Military Officers' School but is given in the Military School for Artillerymen and Engineers, and also in the Military College. The first of these schools is the institution where the officers of the army are trained, the other two being intended for the completion of

the officers' education; the Military College produces officers of the staff. The mathematics taught in these schools concludes with the rudiments of conic sections and the calculus.

In the naval schools there are also courses in mathematics. In the Naval Engineering School, at Yokosuka, where three and one-third years are required for graduation, the following subjects are prescribed: Algebra, geometry, trigonometry, analytic geometry, calculus, elementary dynamics, and applied mechanics. A similar plan for the mathematics courses is followed in the Naval School at Yetejima, where all the Japanese naval officers are trained. In the Naval College at Tokyo some courses in mathematics are also given, K. Ashino being professor of mathematics. The teaching of mathematics in the military and naval schools is sometimes criticised for being much behind the times. The mathematics of the Nautical School of Tokyo is almost the same as in the naval schools.

Finally a few words may be said as to the publication of treatises and periodicals which are undoubtedly contributing largely to the improvement of the knowledge of mathematical studies. Although there is no work worthy of exceptional mention, yet the most interesting is certainly the series of treatises published by T. Hayashi. Books of advanced nature have begun to appear occasionally. We shall not however go into details with respect to Japanese manuals.

Among the mathematical journals published in Japan we may mention the *Journal of the Tokyo Physics School*, the *Mathematical Club*, the *Mathemathecal Journal*, the *Mathematical World*, the *XY*, etc. These are, with the exception of the first named, mostly read by the pupils and graduates of middle class schools. The proceedings of the Tokyo Mathematico-Physical Society contains original contributions, mostly composed in European languages. If one desires to know about the progress of mathematics recently made in Japan, he is requested to consult the mathematical journals. The contributors of mathematical essays may be mentioned: Endo, Fujisawa, Fujiwara, Fukuzawa, Hayashi, Kaba, Kaibara, Kąriya, Kato, Kikuchi, Kimura, Kitao, Kubota, Kumamoto, Mikami, Miwa, Miwada, Mizuhara, Motoda, Naito, Nakagawa, Ogura, Sudo, Sakai, Sawada, Sawayama, Takagi, Terao, Tsuruta, Yoshiye.

The essays of these authors concern various branches of mathematics, some being historical writings.

The Journal of the College of Science, Tokyo, contains also some articles relating to mathematies, the authors being Fujisawa, Nakagawa, Sakai, Sudo, and Takagi.

Besides, the essays contributed by Japanese mathematicians to foreign periodicals, though few in number, are also to be noted. The writings of Fukuzawa and others cannot escape our notice.

The academic degrees are conferred in Japan by the minister of education upon four classes of candidates: (1) those who have successfully

completed their studies in the Graduate Colleges of Imperial Universities, (2) those who have presented their theses, (3) those who have been recommended by the president of the Imperial University from among the professors. T. Takagi is the only mathematician who has ever received his doctor's degree in Japan through the presentation of a thesis. Among the holders of the same degree in the domain of mathematics we may count R. Fujisawa, J. Kawai, D. Kikuchi, the late D. Kitao (learned in mechanics), K. Miwa, and H. Terao (astronomer).

In Japan the publication of articles does not contribute to elevate the author's social position or improve his reputation, and there are many who have never published anything concerning their special studies and yet who have a reputation as learned men and consequently obtain their high positions. This custom will seem very odd to Americans and Europeans, who are full of the spirit of progress. This is certainly a partial reason why the progress of mathematics in Japan has not been as rapid as could be expected or hoped.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

354. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

To find x in the following equation:
$0.002\{6x-20[(1.05)^x-1]\}=0.012\{21[(1.05)^{x-1}-1]-(x-1)\}$.

Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md

The equation readily reduces to

$$3x-35[(1.05)^x-1]=0,$$

from which it is easily seen that $x=0$ is one root of the equation. Another root, we find by a process of approximation, to be $x=21.29$ (nearly).

Also solved by V. M. Spunar, S. G. Barton, and the Proposer.

The equation, being transcendental, has an infinite number of roots, real and imaginary. By compounding the graph of the trandscendental part of the equation with the graph of the linear part, it appears that the above are the only two real roots. Ed. F.

355. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Solve the equations:

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}=a_1;$$
$$\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=a_2;$$
$$\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}=a_3;$$
$$\cdots\cdots\cdots$$
$$\frac{1}{x_{n-2}}+\frac{1}{x_{n-1}}+\frac{1}{x_n}=a_{n-2};$$
$$\frac{1}{x_{n-1}}+\frac{1}{x_n}+\frac{1}{x_1}=a_{n-1};$$
$$\frac{1}{x_n}+\frac{1}{x_1}+\frac{1}{x_2}=a_n.$$

Solution by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

Considering the system of equations as linear in the unknowns $\frac{1}{x_1}$, $\frac{1}{x_2}$, ..., $\frac{1}{x_n}$, the values of the unknowns may be readily found by determinants.

We have for $\frac{1}{x_1}$, the expression,

$$\begin{vmatrix} a_1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ a_2 & 1 & 1 & 1 & 0 & \cdots & 0 \\ a_3 & 0 & 1 & 1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & 1 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & 1 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & 1 & 1 & 1 \\ a_{n-1} & 0 & \cdots & \cdots & 1 & 1 & 1 \\ a_n & 1 & \cdots & \cdots & 0 & 0 & 1 \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 1 & 1 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1 & 1 \\ 1 & 0 & 0 & \cdots & \cdots & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{vmatrix}$$

That is, $\frac{1}{x_1}=\frac{1}{3}\triangle$, where $\triangle$ is the dividend of the above expression, the divisor being equal to 3. When $n\equiv 0 \pmod{3}$, the a's are not independent, and the values of the unknowns, $\frac{1}{x_1}$, $\frac{1}{x_2}$, ..., $\frac{1}{x_n}$, are not definite.

When $n\equiv 1 \pmod{3}$, the numerator of $1/x_1$ is

$$a_1-2a_2+a_3+a_4-2a_5+a_6+a_7-2a_8+a_9+a_{10}-2a_{11}+\ldots$$

$$=\sum_{i=0}^{i=\frac{1}{3}(n-1)} a_{3i+1}+\sum_{j=1}^{j=\frac{1}{3}(n-1)} a_{3j}-2\sum_{k=0}^{k=I[\frac{1}{3}(n-2)]} a_{3k+2},$$

where $I\left(\frac{n-2}{3}\right)$ means the integral part of $\left(\frac{n-2}{3}\right)$.

The values of $\frac{1}{x_2}$, $\frac{1}{x_3}$, ..., $\frac{1}{x_n}$, may be obtained from that of $\frac{1}{x_1}$ by a cyclic permutation of the coefficients with their proper signs.

When $n\equiv 2 \pmod{3}$, the value of the numerator of $1/x_1$ is

$$2a_1-a_2-a_3+2a_4-a_5-a_6+2a_7-a_8-a_9+2a_{10}-\ldots$$

$$=2\sum_{i=0}^{i=I[\frac{1}{3}(n-1)]} a_{3i+1}-\sum_{j=0}^{j=\frac{1}{3}(n-2)} a_{3j+2}-\sum_{k=1}^{k=\frac{1}{3}(n-2)} a_{3k}$$

and the numerators of the other unknowns, $\frac{1}{x_2}$, $\frac{1}{x_3}$, ..., $\frac{1}{x_n}$, may be obtained from that of $1/x_1$, by a cyclic permutation of the coefficients of the a's with their respective signs.

GEOMETRY.

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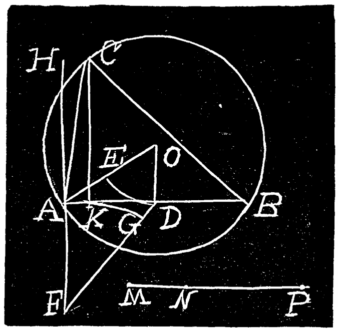
379. Proposed by G. I. HOPKINS, Manchester, N. H.

Construct the triangle, having given base, vertical angle, and ratio of its altitude to difference of other two sides.

Solution by A. H. HOLMES, Brunswick, Maine.

Describe a circle, center O, in which the chord AB (=given base) will subtend the given angle. Draw OD perpendicular to AB, and draw OB. With OD as a radius and O as a center, describe an arc cutting OB in E. Let the divisions MN and NP of the line MP represent the ratio of the difference of the sides of the triangle to the perpendicular. Construct the fourth proportional to MN, NP, and BE, and erect this line, BF, perpendicular to AB at the point B. Draw DF. With F as a center, and FB as a radius, describe an arc cutting FD in G. Find fourth proportional to MN, NP, and $2DG$. Erect this line, BH, perpendicular to AB at point B. Draw HC parallel to AB and cutting circumference in C. Draw AC and BC. Then ABC will be the required triangle, which is shown as follows: $AB=2a$; r, the radius, $=\frac{a}{\cos C}$. Put $x+y$ and $x-y$ for the sides, and p for the ratio of difference of sides to perpendicular. Then we have the proportion, $2r:x+y=x-y:2py$.



$\therefore x^2=y^2+2pry\ldots(1)$.

Also, $\cos C=\frac{(x+y)^2+(x-y)^2-4a^2}{2(x+y)(x-y)}$.

$\therefore (1-\cos C)x^2+(1+\cos C)y^2=2a^2\ldots(2)$.

Eliminating x^2 from (1) and (2), $y^2+2pr(1-\cos C)y=a^2$.

$\therefore y=[a^2+p^2r^2(1-\cos C)^2]-pr(1-\cos C)$.

$\therefore$ Perpendicular $CK=2py=2p\{\sqrt{[a^2+p^2r^2(1-\cos C)^2]}-pr(1-\cos C)\}$.

Also solved by V. M. Spunar.

380. Proposed by W. J. GREENSTREET, A. M., Stroud, England.

$A\ B\ C\ D$ is a quadrilateral, sides in order a, b, c, d, and $B+D=\theta$. Express the diagonals in terms of a, b, c, d, θ.

Solution by A. H. HOLMES, Brunswick, Maine.

Let $ABCD$ be the rhombus, in which $AB=a$, $BC=b$, $CD=c$, $DA=d$. Put $x=AC$, $y=BD$, and $\angle ABC=\psi$. Then $\angle ADC=\theta-\psi$.

In $\triangle ABC$ we have, $\cos\psi=\frac{a^2+b^2-x^2}{2ab}$, and in $\triangle ADC$, $\cos(\theta-\psi)=$

$\frac{c^2+d^2-x^2}{2cd}$. Put $\frac{a^2+b^2}{2ab}=m$, $\frac{1}{2ab}=n$, $\frac{c^2+d^2}{2cd}=p$, and $\frac{1}{2cd}=q$. We find, on eliminating ϕ, and reducing,

$$x^4-2\left[\frac{mn-(mq+np)\cos\theta+pq}{n^2-2nq\cos\theta+q^2}\right]x^2=-\frac{m^2-2mp\cos\theta+p^2-\sin^2\theta}{n^2-2nq\cos\theta+q^2}$$

$$\therefore x=\sqrt{\frac{mn-(mq+np)\cos\theta+pq\pm\sin\theta\sqrt{[n^2-2nq\cos\theta+q^2-(mq-np)^2]}}{n^2-2nq\cos\theta+q^2}}$$

Replacing in this, for m, n, p, and q their equivalents in a, b, c, and d, we have,

$$x=\sqrt{\frac{a^2b^2(c^2+d^2)-abcd(a^2+b^2+c^2+d^2)\cos\theta+c^2d^2(a^2+b^2)\pm abcd\sin\theta\sqrt{[4(a^2b^2-2abcd\cos\theta+c^2d^2)-(a^2+b^2-c^2-d^2)^2]}}{\sqrt{(a^2b^2-2abcd\cos\theta+c^2d^2)}}}$$

For y, we must put b for d in value for x. Also, $-\sin\theta$ for $\sin\theta$, since $\sin(360^\circ-\theta)=-\sin\theta$. Then

$$y=\sqrt{\frac{a^2d^2(b^2+c^2)-abcd(a^2+b^2+c^2+d^2)\cos\theta+b^2c^2(a^2+d^2)\mp abcd\sin\theta\sqrt{[4(a^2d^2-2abcd\cos\theta+b^2c^2-(a^2-b^2-c^2+d^2)^2]}}{\sqrt{(a^2d^2-2abcd\cos\theta+b^2c^2)}}}$$

When $ABC+ADC>180^\circ$, $AC<BD$, and the plus sign is to be used before $abcd\sin\theta$ for x, and the minus sign before the same for y. This is, of course, reversed when $ABC+ADC<180^\circ$.

381. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the number of diagonals of a *complete* polygon of n sides.

I. Solution by J. OWEN MAHONEY, B. E., M. Sc., Central High School, Dallas, Texas.

An incomplete polygon of n sides has $\frac{n}{2}(n-3)$ diagonals. The complete polygon of n sides has $\frac{n}{2}(n-1)$ vertices or $\frac{n}{2}(n-3)$ more than the incomplete polygon of n sides. Hence, the number of diagonals of the complete figure is,

$$\frac{n}{2}(n-3)+\frac{\left[\frac{n}{2}(n-3)\right]\left[\frac{n}{2}(n-3)-1\right]}{2}$$

$$=\frac{n}{2}(n-3)+\frac{n(n-3)(n^2-3n-2)}{8}=\frac{n(n-1)(n-2)(n-3)}{8}.$$

II. **Solution by the PROPOSER.**

A *complete* polygon of n sides is formed by n intersecting lines, and the first question is, how many lines will join the points of intersection of n intersecting lines. n lines can intersect in $\frac{n(n-1)}{2}$ points. From each of these points there can be drawn the other $\frac{n(n-1)}{2}-1$ lines. Therefore, from $\frac{n(n-1)}{2}$ points, $\frac{n(n-1)}{2}\left[\frac{n(n-1)}{2}-1\right]$, of which, evidently, only half can be taken; consequently, the required number of lines is $\frac{1}{8}(n+1)n(n-1)(n-2)$. Since each line drawn from $\frac{n(n-1)}{2}$ points cuts $n-2$ lines, we must subtract $\frac{n(n-1)}{2}(n-2)$ from the last result, in order to obtain the number of diagonals of a complete polygon of n sides. Therefore the required number of diagonals is

$$\tfrac{1}{8}(n+1)n(n-1)(n-2)-\tfrac{1}{2}n(n-1)(n-2)$$
$$=\tfrac{1}{8}n(n-1)(n-2)(n-3),$$

or, more briefly, $3\binom{n}{4}$.

Also solved by Jeannette Brooks and A. H. Holmes.

CALCULUS.

302. **Proposed by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.**

Find an algebraic integral of
$$LCy''+[18L(s-x^2)-2kC]y'+[L(\tfrac{2}{3}k-10x)-8k(s-x^2)]y=0,$$
where $L=2kx+3s-k^2$, $C=12sx-4x^3-t$, $k^4-6sk^2-tk-3s^2=0$. The roots of the determining equation are 0, $-\frac{1}{2}$ for a root of $C=0$; 0, 2 for a root of $L=0$; and 1 and $\frac{3}{2}$ at infinity.

No solution of this problem has been received.

303. **Proposed by C. N. SCHMALL, New York City.**

If $A=\int_0^1\frac{dx}{\sqrt{(1-x^4)}}$, express A in terms of Gamma-functions. [Part of ex. 39, p. 474, Bromwich's *Theory of Infinite Series*.]

I. **Solution by JOSEPH A. NYBERG, Student, University of Chicago.**

In the integral

(1) $$u=\int_0^1 x^{\lambda-1}(1-x^{\mu})^{\nu-1}dx$$

make the substitution, $x^{\mu}=y$. Then

$$dx=\frac{1}{\mu}y^{(1/\mu)-1}dy,\quad \begin{cases} x=0,\ \ldots,\ 1\\ y=0,\ \ldots,\ 1\end{cases}$$

Therefore,

(2) $$u=\frac{1}{\mu}\int_0^1 y^{(\lambda/\mu)-1}(1-y)^{\nu-1}dy\equiv\frac{1}{\mu}B\left(\frac{\lambda}{\mu},\nu\right)$$

by the definition of the beta function. Using the formula,

(3) $$B(a,\,b)=\frac{\Gamma(a)\ \Gamma(b)}{\Gamma(a+b)}$$

we get

(4) $$u=\frac{1}{\mu}\,\frac{\Gamma(\lambda/\mu)\ \Gamma(\nu)}{\Gamma[(\lambda/\mu)+\nu]}.$$

(4) is the solution of (1) in terms of gamma functions.

On putting $\nu=\frac{1}{2}$, $\lambda=1$, $\mu=4$, we have

$$\int_0^1\frac{dx}{\sqrt{(1-x^4)}}=\frac{1}{4}\,\frac{\Gamma(\frac{1}{4})\ \Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}$$

Proofs of the formula,

$$B(a,\,b)=\frac{\Gamma(a)\ \Gamma(b)}{\Gamma(a+b)}$$

are found in Byerly's *Integral Calculus*, p. 114; in an article by A. Pringsheim in *Mathematische Annalen*, Band 31 (1888), p. 480; and in N. Nielsen's *Hankbuch der Theorie der Gamma Funktion*, p. 148.

II. **Solution by FRANCIS RUST, C. E., Pittsburg, Pa.**

Let $x^2=\sin\phi$. Then $dx=\dfrac{\cos\phi\ d\phi}{2\sqrt{(\sin\phi)}}$. Hence,

$$A=\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\sin^{-\frac{1}{2}}\phi\,\cos\phi\,d\phi=\tfrac{1}{4}B(\tfrac{1}{4},\,\tfrac{1}{2})=\tfrac{1}{4}\frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}=\tfrac{1}{4}\sqrt{(\pi)}.\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}.$$

In elliptic integrals, using Legendre's notation, $A=\frac{1}{2}\sqrt{(2)}F^I(\frac{1}{2}\sqrt{2})=1.311029$.

Also solved similarly by James A. Whitted, and the Proposer.

304. Proposed by H. C. FEEMSTER, York College, York, Neb.

Reduce $axyp^2+(x^2-ay^2-b)p-xy=0$ to Clairaut's form, and hence solve the equation.

Solution by OLLINE CRUICKSHANKS, Bowling Green, Ky., and S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, New York.

Expanding and factoring,

$$apy(px-y)+x(px-y)=bp,\quad (apy+x)(px-y)=bp. \tag{I}$$

Let $x^2=u$ and $y^2=v$. $\therefore 2xdx=du$ and $2ydy=dv$.

$$p=\frac{dy}{dx}=\frac{x}{y}\,\frac{dv}{du}.$$

Substituting this value of p in (I),

$$\left(\frac{x^2}{y}\,\frac{dv}{du}-y\right)\left(\frac{axy}{y}\,\frac{dv}{du}+x\right)=\frac{bx}{y}\,\frac{dv}{du}. \tag{II}$$

Multiplying (II) by y/x,

$$\left(x^2\,\frac{dv}{du}-y^2\right)\left(a\,\frac{dv}{du}+1\right)=b\frac{dv}{du}. \tag{III}$$

Since $x^2=u$ and $y^2=v$, (III) becomes

$$(u\frac{dv}{du}-v)(a\frac{dv}{du}+1)=b\frac{dv}{du}. \tag{IV}$$

$$\therefore v=u\frac{dv}{du}-\frac{b\dfrac{dv}{du}}{a\dfrac{dv}{du}+1}. \tag{V}$$

$\therefore v=uc-\dfrac{bc}{ac+1}$; but $v=y^2$ and $u=x^2$. $\therefore y^2=cx^2-\dfrac{bc}{ac+1}$.

Also solved by A. H. Holmes, who substitutes for p, xz/y, and then differentiates the resulting equation.

305. Proposed by C. N. SCHMALL, New York City.

Prove $\int_\beta^x \frac{dx}{\sqrt{[(\alpha-x)(x-\beta)]}} = 2\cos^{-1}\sqrt{\frac{\alpha-x}{\alpha-\beta}}$. [Edwards' *Integral Calculus for Beginners*, p. 84, ex. 4.] Does this result hold when the upper limit is changed from x to α?

Solution by A. M. HARDING, Associate Professor of Mathematics. University of Arkansas.

Put $x=\beta\cos^2\theta+\alpha\sin^2\theta$, *i. e.*, $\theta=\cos^{-1}\sqrt{\frac{\alpha-x}{\alpha-\beta}}$; then $\alpha-x=(\alpha-\beta)\cos^2\theta$, $x-\beta=(\alpha-\beta)\sin^2\theta$, $dx=(\alpha-\beta)\sin 2\theta\, d\theta$.

Hence, $\int_\beta^x \frac{dx}{\sqrt{[(\alpha-x)(x-\beta)]}} = \int_0^\theta \frac{(\alpha-\beta)\sin 2\theta\, d\theta}{(\alpha-\beta)\sin\theta\cos\theta}$

$$= 2\int_0^\theta d\theta = 2\theta = 2\cos^{-1}\sqrt{\frac{\alpha-x}{\alpha-\beta}}.$$

When the upper limit is α, we have

$$\int_\beta^\alpha \frac{dx}{\sqrt{[(\alpha-x)(x-\beta)]}} = \underset{\varepsilon \doteq 0}{\text{Lim.}} \int_\beta^{\alpha-\epsilon} \frac{dx}{\sqrt{[(\alpha-x)(x-\beta)]}}$$

$$= \underset{\varepsilon \doteq 0}{\text{Lim.}}\, 2\cos^{-1}\sqrt{\frac{\alpha-(\alpha-\varepsilon)}{\alpha-\beta}} = 2.\tfrac{1}{2}\pi = \pi.$$

Also solved in various ways by C. E. Rust, A. H. Holmes, V. M. Spunar, and the Proposer.

MECHANICS.

253 (Incorrectly numbered 354). Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

On a smooth horizontal plane are touching each other two balls, equal, uniform, inelastic, spherical, radius x, mass M. A third uniform, smooth, elastic ball, radius y, mass m, is placed with its centre vertically above the point of contact of those balls. Its radius may vary as long as its mass remains m. When is the velocity of each inelastic ball after impact a maximum?

No solution of this problem has been received.

354 (Incorrectly numbered 355). Proposed by the late G. B. M. ZERR, Ph. D.

Find the current given by a battery $\frac{1}{2}n(n+1)$ cells arranged in n rows as follows: First row, 1 cell; second row, 2 cells; third row, 3 cells; nth row, n cells. The electromotive force of each cell is E and its resistance r, while the resistances of the wires joining the positive and negative ends of the rows to the poles are as follows: First row, nr; second row, $(n-1)r$; third row, $(n-2)r$, ..., $(n-2)$nd row, $3r$; $(n-1)$st row, $2r$; nth row, r. The resistance of the two wires joining the poles to complete the circuit is r for each.

Solution by V. M. SPUNAR, C. E., Chicago, Ill.

If the cells in the mth row are connected in series, the current in that row is, by Ohm's law,

$$C_m=\frac{mE}{mr+R_m},$$

where R_m is the external resistance$=2r(n-m+1)$, by the conditions of the problem. Hence, the total current is

$$C=\sum_{m=1}^{m=n}\frac{mE}{mr+2r(n-m+1)+2r}=\frac{E}{r}\sum_{m=1}^{m=n}\frac{m}{2(n+2)-m}$$

$$=\frac{E}{r}\sum_{m=1}^{m=n}\left[-1+\frac{2(n+2)}{2(n+2)-m}\right]=\frac{E}{r}\left[-n+2(n+2)\sum_{m=1}^{m=n}\frac{1}{2(n+2)-m}\right]$$

$$=\frac{E}{r}\left[2(n+2)\left(\frac{1}{2n+3}+\frac{1}{2n+2}+\frac{1}{2n+1}+\ldots+\frac{1}{n+4}\right)-n\right]$$

$$=\frac{E}{r}\left[2(n+2)\,(S_{2n+3}-S_{n+3})-n\right],\quad \text{where } S_p=\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{p}.$$

But $S_{2n+3}-S_{n+3}=\log(2n+3)-\log(n+3)$

$$+\frac{1}{2(2n+3)}-\frac{1}{2(n+3)}-\frac{1}{12(2n+3)^2}+\frac{1}{12(n+3)^2}\ldots,$$

(see MONTHLY, Vol. XVI, p. 166)

$$=\log\left(\frac{2n+3}{n+3}\right)+\tfrac{1}{2}\cdot\frac{n}{(n+3)(2n+3)}+\tfrac{1}{4}\cdot\frac{n(n+2)}{(n+3)^2(2n+3)^2}+\ldots$$

When $n\doteq\infty$, $S_{2n+3}-S_{n+3}\doteq\log 2$.

NOTE. Contributors are asked to send in solutions of problems 255, 256, 257, 258, and 259. 255 is on page 247, Vol. XVII.

AVERAGE AND PROBABILITY.

203. Proposed by J. EDWARD SANDERS, Weather Bureau, Columbus, Ohio.

Find the average length of a hole at random through a cone.

No solution of this problem has been received.

204. Proposed by F. P. MATZ, Reading, Pa.

On a random chord in a circle two points are taken at random. What is the chance a second chord drawn at random will pass between the two points?

Solution by the late G. B. M. ZERR, Ph. D.

Let M, N be the two random points on the random chord AB; O, the center of the circle; $OA=OB=r$; $AM=y$; $MN=x$; $\angle AOB=2\theta$; μ the angle AB makes with some fixed line. Then $AB=2r\sin\theta$; an element of the circle at M is $r\sin\theta\, d\theta\, dy$; at N, $d\mu\, xdx$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of μ, 0 and 2π; of y, 0 and $2r\sin\theta$; of x, 0 and y. If the second random chord intersects the first, the chance that it passes between the points is $x/2r\sin\theta$. The chance that the second chord intersects the first is $\frac{1}{3}$. Therefore, if p=the chance, we get

$$p=\frac{\frac{1}{3}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\int_0^{2r\sin\theta}\int_0^y rx^2\sin\theta\, d\theta\, d\mu\, dy\, dx}{\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\int_0^{2r\sin\theta}\int_0^y 2r^2x\sin^2\theta\, d\theta\, d\mu\, dy\, dx}=\frac{1}{6}.$$

Also solved by J. E. Sanders, who obtains 3/16 as a result.

MISCELLANEOUS.

177. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Sum the infinite series:

(a) $\sin x+nx\cos x-\dfrac{n^2x^2}{2!}\sin x-\dfrac{n^3x^3}{3!}\cos x+\dfrac{n^4x^4}{4!}\sin x+\ldots,$

(b) $\cos x-nx\sin x-\dfrac{n^2x^2}{2!}\cos x+\dfrac{n^3x^3}{3!}\sin x+\dfrac{n^4x^4}{4!}\cos x\ldots$

Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md., and G. B. M. ZERR, Ph. D.

(a) Arranging, we get

$$\sin x\left[1-\frac{n^2x^2}{2!}+\frac{n^4x^4}{4!}\ldots\right]+\cos x\left[nx-\frac{n^3x^3}{3!}+\frac{n^5x^5}{5!}\ldots\right]$$

$=\sin x\cos nx+\cos x\sin nx=\sin(n+1)x.$

$$\text{(b)}\quad \cos x\left[1-\frac{n^2x^2}{2!}+\frac{n^4x^4}{4!}\ldots\right]-\sin x\left[nx-\frac{n^3x^3}{3!}+\frac{n^5x^5}{5!}\ldots\right]$$

$=\cos x\cos nx-\sin x\sin nx=\cos(n+1)x.$

Also solved by Francis Rust and V. M. Spunar.

178. Proposed by V. M. SPUNAR, Mechanical and Civil Engineer, East Pittsburg, Pa.

Find the sum of the series, $\frac{\sin x}{m^2+1}-\frac{2\sin 2x}{m^2+4}+\frac{3\sin 3x}{m^2+9}-\ldots$ to infinity.

Solution by G. B. M. ZERR, Ph. D.

From the general equation in Fourier's Series, we at once derive

$$\frac{\pi}{2}\sin nx=\sin x\int_0^\pi \sin nx\sin x\,dx+\sin 2x\int_0^\pi \sin nx\sin 2x\,dx$$

$$+\sin 3x\int_0^\pi \sin nx\sin 3x\,dx+\ldots$$

But $\int_0^\pi \sin nx\sin rx\,dx=\pm\frac{r\sin n\pi}{r^2-n^2}$, according as r is odd or even.

$$\therefore\ \frac{\pi\sin nx}{2\sin n\pi}=\frac{\sin x}{1-n^2}-\frac{2\sin 2x}{4-n^2}+\frac{3\sin 3x}{9-n^2}-\ldots$$

Let $n=m\sqrt{-1}$.

$$\therefore\ \frac{\sin x}{m^2+1}-\frac{2\sin 2x}{m^2+4}+\frac{3\sin 3x}{m^2+9}-\ldots=\frac{\pi\sinh mx}{2\sinh m\pi}=\frac{\pi}{2}\cdot\frac{e^{mx}-e^{-mx}}{e^{m\pi}-e^{-m\pi}}.$$

NOTES AND NEWS.

Professor Jose A. Caparo, of Notre Dame University, Notre Dame, Indiana, has been given a year's leave of absence, and will spend most of his time in Cuzco, Peru, South America. F.

Before leaving for California, Editor Slaught was called to New York State by his Alma Mater, Colgate University, to receive the honorary degree of Doctor of Science. He graduated there in 1883, and has been engaged in educational work ever since. The degree of Doctor of Philosophy was conferred upon him by the University of Chicago, in 1898, in which institution he is now an Associate Professor of Mathematics. He is the author, in collaboration with Dr. Lennes, of Columbia University, of a series of text books in mathematics for secondary schools, which are being very extensively adopted throughout the country. We congratulate Dr. Slaught on the honor conferred upon him by his Alma Mater. F.

Editor Slaught is in California attending the meeting of the National Education Association as the representative of the University of Chicago. He will present the report of the National Committee of Fifteen on Geometry at the meeting of the Secondary Department. This committee has been working for two years under the joint auspices of the National Education Association and the American Federation of the Physical and Mathematical Sciences. The report in provisional form was published in the April, May, and June issues of *School Science and Mathematics* and was distributed in the form of reprints to two hundred selected critics for further suggestions and criticisms before the final presentation at San Francisco. The report contains a valuable historical introduction by Professor Florian Cajori, which was initially published in THE MONTHLY for November, 1910. Then follow sections on logical considerations, special courses in geometry, exercises and problems, and finally the syllabus itself, which sets forth a scheme for emphasizing the great basal theorems of geometry, in comparison with those of subsidiary importance, by means of various styles of type. ED. F.

The following doctors and candidates for the doctorate in mathematics from the University of Chicago, have been appointed to positions for 1911-1912, as indicated below:

Professor F. L. Griffin, 1907, of Williams College, to the professorship of mathematics at Reed Institute, Portland, Oregon.

Dr. H. E. Buchanon, 1909, of the University of Wisconsin, to the professorship of mathematics at Carleton College, Northfield, Minnesota.

Dr. E. J. Miles, 1910, of Cornell University, to an instructorship in mathematics at Yale University.

Dr. Anna J. Pell, 1910, to an instructorship in mathematics at Mt. Holyoke College.

Mr. Lloyd Dines to an instructorship in mathematics at Columbia University.

Dr. D. Buchanon, 1911, to an assistant professorship in mathematics at Queen's University, Kingston, Ontario.

Mr. R. E. Root to an instructorship in mathematics at the Univesity of Missouri.

Messrs. Dines, and Root are finishing their work during the present summer quarter. S.

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THE TEACHING OF MATHEMATICS IN SUMMER SESSIONS OF UNIVERSITIES AND NORMAL SCHOOLS.*

By H. E. SLAUGHT, The University of Chicago.

THE SUMMER SESSIONS IN GENERAL.

Previous to the year 1891 there had been no general movement in this country to organize summer courses in connection with the schools, colleges, and universities. Certain summer courses for teachers had been offered at Harvard University as early as 1871, but not until 1891 were any regular courses in mathematics given. In 1891 also certain courses in mathematics and latin were offered in a summer session at the State Normal School at Emporia, Kansas, as a private undertaking of two instructors in the school. In like manner some summer courses were offered in the same year at the University of California, the University of Indiana, and the University of Minnesota. No doubt, complete data would show that similar beginnings were made about this time, or not much later at other institutions. Cornell University made a start in 1892, the University of Missouri and the University of Illinois in 1894, and the University of Texas in 1897. In most of these cases, if not all, the beginnings were made by a few instructors who carried on the work as a semi-private undertaking, depending wholly upon the tuition receipts for their remuneration.

The University of Chicago was the first institution to embody in its constitution a provision for a regular summer session conducted by the permanent faculty under conditions precisely similar to those in the other sessions of the school year. President Harper maintained that to allow a great and costly University plant to stand idle during one quarter of the year was neither good business nor good educational policy. He held that the advantages of college and university training should not be kept forever out of reach of those who were engaged in teaching and who could not afford to

* This paper was prepared on short notice at the request of the American Commissioners of the International Commission on the Teaching of Mathematics. They discovered at the last moment that this phase of work for the preparation of teachers had been overlooked in the assignment of jurisdiction to other committees. As there was not time for an exhaustive investigation, it was decided to gather data from a limited number of representative institutions and to draw such conclusions as might be possible on this basis.

give up their positions in order to attend the university during the nine months from October to June. How keen was his prophetic vision and how great a boon he conferred upon the teachers of the country may be inferred from the data of attendance shown in the tables of statistics given below. No summer session was held at Chicago during the first year, 1892-3, on account of the Columbian Exposition in close proximity to the campus, but the summer quarter of 1894 was attended by 597 students, and the enrollment for the next seven summers was as follows: 931, 1048, 1273, 1434, 1636, 2375.

The steadily growing attendance at these summer sessions, the serious character of the work done, the extent and variety of the courses offered both for elementary and advanced students, the devotion to this cause of the regular university faculty on exactly the same terms as at any other session of the year,— these were new phenomena in the educational development in this country which may well be characterized as the beginning of an epoch; namely, the adoption of the summer session as a part of the regular program in many colleges and universities.

It cannot, however, be said that this program has been, or is likely to be universally adopted by the colleges and universities. Indeed, in the nature of the case, only the larger institutions, and especially the State universities, are likely to develop this phase of work strongly. Inasmuch as the constituency of the summer session is very largely from the ranks of teachers, it is natural that those institutions which have developed strong departments, or schools of education, should attract the larger number. One by one these institutions have organized the work of the summer session; usually at first in a tentative way, being in charge of a few instructors who might assume the responsibility and derive such revenue as the tuition fees would afford; later receiving official endorsement both as to financial responsibility and in the granting of credit toward degrees for work done in these courses. In many cases it is only recently that full credit for summer work has been accorded toward the Bachelor's degree, and in very few instances even yet can credit toward a higher degree be obtained, as at Chicago and Wisconsin. Columbia for example, explicitly provides that credit for work in the summer sessions may count toward the Bachelor's or Master's degree but not toward the Doctor's degree. The extension of the credit privilege corresponds closely to the gradual development of the summer programmes. Naturally the beginnings were made in most cases with elementary courses only, and later advanced courses were gradually added. Chicago furnishes a notable exception to this manner of development. There the full curriculum of courses, elementary and advanced, could be found from the very outset and all degrees were included in the range for which credit could be received.

The following table shows certain general data concerning the summer sessions of eighteen universities which are sufficiently representative to indicate the extent to which this phase of work has gained recognition.

TABLE I.

Institutions.	Date of Organizing Summer Courses.	Number of Weeks in the Summer Sessions.	Attendance for past five years: 1906.	1907.	1908.	1909.	1910.
California	1891	6	706	522	661	819	1051
Chicago	1894	12	2688	2605	3050	3264	3370
Columbia	1900	6	1043	1392	1998	1946	2629
Cornell	1892	6	642	755	841	889	987
Harvard	1871	6	779	809	1116	903	873
Illinois	1894	9	423	502	555	631	665
Indiana	1890	12	682	721	1005	1139	1107
Iowa	1900	6	281	344	363	363	347
Kansas	1903	6	264	289	377	374	390
Michigan	1894	8	1034	1070	1085	1224	1234
Minnesota	1891	6	1019	1035	1218	1286	1188
Missouri	1894	8	403	542	508	552	576
Oberlin	1899	8	142	136	142	160	154
Ohio State	1905		385	425	508	642	639
Pennsylvania	1904	6	505	707	879	826	1219
Texas	1899	7	444	580	625	741	848
Washington State	1904	6	198	243	235	288	303
Wisconsin	1899	6	650	750	1025	1125	1253

Remarks on Table 1. At the University of California, summer courses have been given since 1891, but the first regular summer session was held in 1900, since which time full credit toward the baccalaureate degree has been allowed for all summer courses of college grade. This is typical of many other institutions, for instance, at the University of Illinois, summer courses were given since 1894, but the regular sessions granting credit for work of a college grade began in 1900.

The figures for the University of Minnesota and for the University of Texas include those registered for elementary work as well as those taking work of college grade. At the University of Texas the summer session is combined with a summer normal school. This arrangement prevails also in some other cases, for instance, at Tulane University, where the summer session was introduced in 1910.

MATHEMATICS COURSES IN THE SUMMER SESSIONS.

The foregoing general statement with reference to the development of the summer sessions in American universities forms an appropriate background for the consideration of the work of mathematics in particular in these sessions. It appears that the courses in mathematics have been regarded in most cases as of special importance in these summer curricula. In fact, in several instances the work began with courses in mathematics and perhaps no other subjects, and was gradually extended to include courses in all the regular departments as the number of students increased and the demand became apparent.

The courses offered in mathematics may be roughly divided into two

classes: (1) those intended primarily to emphasize the pedagogical aspect of the subject, and (2) those intended primarily to develop the subject matter for its own sake or as a prerequisite to other courses. The pedagogical courses include critical studies of the various elementary and secondary branches with reference to scientific interpretation and methods of presentation, and also studies in the history of mathematics with special reference to the needs of teachers in elementary and secondary schools. As yet there appear to have been no pedagogical courses offered for the special benefit of teachers of college mathematics, but it may be said that practically every course offered in the summer session becomes in a certain sense a pedagogical course, since so large a proportion of the students are teachers who are keenly alive to their opportunities for observation. In fact it is a common occurrence in these summer courses for students to register for the purpose of studying the teacher quite as much as for studying the subject. Moreover, it is believed by many that the best pedagogical training comes through careful and diligent study of the subject matter, under the guidance of an inspiring teacher who knows how to exhibit good methods and to impress them upon the students by example, rather than by precept. However, there is undoubtedly a growing tendency to attach greater importance to definite, scientific study of the art of teaching, and this tendency is shown in the increasing number of pedagogical courses offered in colleges and universities, especially in the summer sessions, and in this respect mathematics is well represented. Naturally such courses appear most strongly in those institutions in which the department of education, or the separate school of education, is most fully developed.

Of the courses intended primarily for the study of subject matter, comparatively few extending beyond elementary calculus are offered in summer sessions, except in one or two institutions, notably at Chicago and Wisconsin. This may be accounted for in two ways: (1) because in most cases the great majority of students in the summer sessions are not prepared for advanced work, and (2) because the summer school usually has no endowment and the receipts from tuition fees are not sufficient to provide the more expensive iustruction in advanced courses where the enrollment is comparparatively small. The following statement from the former director of the Harvard summer school is fairly typical of the way in which the mathematical work has developed in the summer sessions in other institutions: "Mathematical courses were first introduced into the Harvard summer school in 1891. Courses in solid geometry, plane trigonometry, and college algebra have been given regularly for the past twenty years. Courses in analytic geometry have been given regularly for the past ten years. Courses in calculus regularly for the past seven years. In 1903 a course was given in the theory of functions, and in 1909 a course in modern methods in geometry. In 1910 the following advanced courses were offered: Introduction to modern geometry, topics in the theory of functions, each with very small registration."

The following table shows certain data with respect to mathematics in summer sessions.

TABLE II.

Institutions.	Registration in Mathematics Courses 1906	1907	1908	1909	1910	Elementary Courses including Calculus	Advanced Courses beyond Calculus	Pedagogical Courses
California	41	38	63	60	108	4	1	1
Chicago	286	316	350	380	493	12	4	3
Columbia	130	160	170	210	275	7	3	2
Cornell	Not reported			369	375	8	2	2
Harvard	58	54	43	37	40	5	2	1
Illinois	110	136	152	183	164	5	4	1
Indiana	98	150	177	173	189	4	2	—
Iowa	46	80	73	68	61	7	—	2
Kansas	13	21	13	20	39	3	1	—
Michigan	Average of 155 per year					6	4	1
Minnesota	Average of 100 per year					3	—	—
Missouri	95	98	56	79	85	5	1 or 2	1 or 2
Nebraska	33	33	30	32	42	4	—	—
Oberlin	Average of 35 per year					4	—	1
Ohio State	Not reported				113	4	1	1
Pennsylvania	88	129	151	121	184	11	—	1
Texas	348	420	454	446	342	3	—	1
Washington State	27	29	19	22	22	1	—	2
Wisconsin	142	117	249	225	242	9	4	—

Remarks on Table II. The data at hand do not make it clear in all cases whether the figures in the above table indicate the total registration or the actual number of different students, as in the case of Iowa and Kansas and Oberlin. The numbers for Texas include attendance upon mathematics courses in a summer normal school which is held in connection with the summer school of the university.

At the University of Pennsylvania no advanced course in mathematics was offered in the summer of 1910, though three or four courses had been offered up to that time. Also the elementary courses include some preparatory subjects which will not be given hereafter. (The data as to courses in all cases refer to 1910 unless otherwise specified.)

At the University of Iowa, while no advanced courses are scheduled, yet those students in the summer sessions who desire some advanced work are cared for in small groups as far as possible.

At the University of Kansas no advanced courses have hitherto been offered. Two are to be given in 1911.

The registration in mathematics at Indiana University and at the University of Chicago is given for the first six weeks of the summer sessions. The attendance is considerably diminished during the second six weeks. These are the only institutions reported in which the session consists of two terms of six weeks each.

At the University of Illinois, previous to 1910, courses in preparatory subjects were offered. The registration given in the table is exclusive of these courses.

The Training of Teachers.

In general it may be said that a large number of those in attendance at the summer sessions are teachers spending a part or all of their vacation in study, but it does not follow that all are seeking the strictly pedagogical courses. In fact, very many are pursuing courses for degrees and are therefore filling out requirements or choosing elective work in subject matter in which they are interested. It is true, however, that in most cases where special courses of a pedagogical character are given they are well attended and fully appreciated. On the other hand, there is a wide difference of policy, among the institutions offering summer work, as to the usefulness of the pedagogical courses as compared with the content courses.

The following comments from reports of various institutions in the above list will illustrate these points with respect to the department of mathematics.

"As seen from our enrollment the interest in summer school mathematics is not very great, most teachers preferring work which does not demand so great a strain as the study of mathematics. Whatever work is done has a decidedly beneficial influence upon those who take it."

"Only one course in the pedagogy of mathematics has been offered and this has not been largely attended. It is not possible to form any estimate as to the influence of this work in general upon the preparation of teachers of mathematics."

"A course in algebra has been given from the first, arranged with special reference to broadening the point of view of the high school teacher. Graphical methods are developed, the theory of complex numbers presented and the notation of determinants explained. For the last four summers a course in the history of mathematics has been offered, in which the development of geometry and algebra is traced as carefully as may be from the earliest times. A large proportion of the students in mathematics are enrolled in one or the other of these two courses for teachers and their obvious interest in, and profit from, the work has been gratifying. It is believed that the direct influence of these summer courses upon the school teachers, by far the most numerous class of students in the session, is greater than that of any other work which we do during the academic year."

"Of the advanced courses given in 1910, two were special courses intended for the training of teachers in mathematics. In the one year of their existence these two special courses for teachers attracted many students who showed much interest in the work. It is the estimate of the department of mathematics here that the summer sessions have had good influence upon the preparation of teachers of mathematics."

"In general, the demand for the advanced courses in the summer sessions tends to increase while that for the most elementary courses is stationary or decreasing. Many courses, especially those of intermediate character on such subjects as modern geometry, projective geometry, and selected topics in algebra, are especially adapted to teachers. Some of these courses are very stimulating to them and show them how to use the literature and text books in their subjects more adequately. Not a few of the teachers in attendance in mathematics in the summer sessions drift later into the university."

"The special courses intended for the training of teachers in mathematics have been (1) a general course covering the usual high school work, and (2) conferences. It should be added, however, that great emphasis is laid on those aspects of all the courses which are specially helpful to teachers. The interest in these courses manifested by teachers and persons preparing to teach has been distinctly good. We think the summer work has had a decidedly beneficial influence on the preparation of teachers in mathematics. It seems repeatedly to have led teachers to enter the University for the purpose of completing their college courses. We have frequently heard encouraging remarks from teachers concerning the benefit they have derived from their summer work. Every thing considered, the summer session seems to have had even a greater influence on the efficiency of teaching than the regular sessions."

"No special courses in mathematics for the training of teachers are offered during the Summer, this work being given during the regular year."

"No course in mathematics is presented for the special purpose of teaching the students how to teach the subject, but frequent suggestions are given to that end. Most teachers who take them do so for the subject matter, as well as to observe how those in charge of the courses present them."

"All those in the course for teachers take a great interest in the subject and do much collateral reading in our extensive pedagogical library in mathematics. All express themselves as greatly benefitted by the course. Many in the course have had from three to ten years' experience in teaching."

"During the past ten years a course in the history and teaching of elementary mathematics has been given about five different times. Interest in these several courses intended for the training of teachers of mathematics, on the part of college students, has been very slight, registration in them being composed mainly of teachers in service. The influence of the summer courses upon the preparation of teachers of mathematics has been very helpful to the rather limited number of teachers who have attended the courses."

"The courses in mathematics, especially the one for the training of teachers, have called forth a good deal of interest and have been very valu-

able and inspiring. Even more satisfactory is the teachers' course which we carry throughout the college year, enrolling both those who have taught mathematics and those who are looking toward teaching in secondary schools. We are convinced that the teaching efficiency of these students is greatly increased by reason of their facing in this way the various problems that they will meet at the outset of their teaching experience."

It may be noted that the establishment of well organized schools of education, on the same basis as other professional schools, in some of the leading universities has had a marked influence upon the standards which the public schools are beginning to prescribe with respect to the preparation of their teachers, and wherever these professional schools for teachers hold summer sessions the attendance is large and increasing and the interest is rapidly growing. Some states are already prescribing by statute a minimum of work in education which must be fulfilled by all candidates for high school positions. One of the leading states in this respect is California, where a year of post-graduate work in education in the state university, or in a university of equally high standing, is demanded of all high school teachers. The work in the summer session is allowed to count toward fulfilling this requirement. This will explain the following extract from the California report. "The majority of the students in the summer sessions are from the high schools of the State, with an increasing number of graduates of other institutions who are preparing to teach in our high schools. We require a year of graduate work for the high school certificate, and have recently agreed to accept summer session residence at the rate of four summer sessions for one year of the regular session. We find considerable interest among our best high school teachers in the summer session courses, and believe we are able to help them to a broader view of the subjects which they have to teach. The demand which they make on us for these courses is increasing every year, and is larger than we are able to meet for advanced work, owing to the fact that the administration still insists that the summer session must pay for itself."

Another feature of growing significance is the fact that the number of men, in proportion to the women, is steadily increasing in the professional schools of education, as is not the case in the ordinary normal school. This is particularly noticeable in the summer sessions of the larger universities where the professional side of education is most largely developed. This seems to indicate that the demand for teachers, and particularly for men, with professional training is increasing, and teachers everywhere are beginning to recognize the necessity of such training in order to secure advancement in rank and salary. The first step is naturally the summer school and this explains the rapidly increasing attendance upon these sessions all over the country, and by all classes of teachers from the elementary schools to the high schools and colleges.

Summer Sessions in Normal Schools.

Reports have been received from a limited number of representative normal schools in various parts of the country. In a general way it may be said that the normal schools in the East and in the far West do not hold summer sessions, and that those in the middle West do hold summer sessions, often extending through twelve weeks. There are important exceptions to this statement, and of course, there are many institutes and various kinds of summer schools for teachers which provide lectures and other forms of instruction through periods of from one to four weeks, but which are not here classed with those summer sessions which provide continuations of the regular work in normal schools.

Extracts from a few replies: "The two normal schools in New Jersey hold no summer session since their services are not needed in view of the large opportunities offered in the universities of New York."

"The only normal school in Massachusetts which makes much of summer work is that at Hyannis."

"The normal school at Danville, Conn., has held a four weeks session during July, beginning in 1908. Its object is to give to teachers of common schools who consider themselves unable to take the regular two years' course a brief summary of the common school subjects, what to teach and how to teach it."

"We have in Maryland two normal schools, neither of which holds a summer session."

"The State normal school at Los Angeles, California, has never held summer sessions. Generally one of the five normal schools in the state holds a summer session, but none will be held this year (1911)."

"The two state normal schools of Ohio, one at Oxford and one at Athens, hold summer sessions. The former began the Summer work in 1903 and has had an attendance in each of the last five years ranging from 550 to 700. The registration in mathematics classes has averaged about 170. For the session of 1911 nine courses in mathematics are offered, from arithmetic to trigonometry, and including two courses on the teaching of elementary mathematics."

"The state normal school at Kirksville, Missouri, has held four sessions of twelve weeks during each of the past eight years. The attendance during the summer term has varied during the past five years from 500 to 610. The estimated registration in mathematics courses each year has been about sixty per cent of the total attendance. Courses of both high school and college rank are offered from elementary algebra to calculus, including two courses in the teaching of mathematics. The influence of our mathematics courses has been to induce many students to return to the institution summer after summer for mathematics and other subjects. Many of our mathematics students discover, while here, their ability in mathematics, and develop a desire to be specialists in the subject."

"All of the Michigan state normal schools hold summer sessions. The one at Kalamazoo began on this plan at its organization in 1904. The attendance during the past seven years has increased from 117 in 1904 to 834 in 1910. The courses offered in mathematics below calculus are college algebra, trigonometry, elementary algebra, and geometry. A teachers' course in arithmetic and review arithmetic are intended for the training of teachers of mathematics. Regarding the influence of this work upon the preparation of teachers of mathematics, I will say that very great interest has been shown, especially on the part of those preparing for the grades; also many have taken work looking forward to teaching high school mathematics, and a number of graduates have continued work in the universities."

"All of the Wisconsin state normal schools hold summer sessions. The one at Platteville began two years ago with a session of six week's duration, and last year (1910) had an attendance of 265, of whom about 160 were registered in elementary mathematics courses. We have little opportunity as yet to judge as to the effect of these courses upon the preparation of teachers of mathematics."

All of the normal schools of Illinois hold summer sessions and at least one, at Normal, Illinois, has a session of twelve weeks. All these sessions are largely attended.

"The state normal school at Emporia, Kansas, has had summer sessions in some form since 1891. Mathematics and latin were the only subjects in which instruction was given at that time. The summer session soon developed into a regular feature of the institution and for a number of years past has been organized upon exactly the same basis with respect to employment of faculty, records, and requirements, as at any other portion of our school year. It constitutes one of the five terms of the year. In other words, we do in the summer school the work of one-half a semester,— nine full weeks of class work. The courses offered in mathematics have been the regular work of the normal school in arithmetic, algebra, and geometry of secondary grade; and in higher algebra, trigonometry, surveying, analytics, and calculus of college grade. The other normal schools of the state, one of them eight years old, the other ten, have had summer sessions on the same basis ever since they were organized. The attendance at Emporia, in 1910 was 1200, and the number of people enrolled in mathematics courses was 305. During the past five years the average attendance has been about 1000, the registration in mathematics approximately 250. We draw from all classes of teachers in the state including many who are doing high school work, and the general influence upon the teaching work of the state has been quite extensive. It is difficult to give any accurate or specific data with regard to this, but in all of the courses offered in mathematics it has been the intention to consider them as specially organized with reference to preparation for teaching."

Enough has been said to indicate the general influence of the normal

school summer sessions, especially in the middle West. It is clear that for teachers, especially in the common schools and in the elementary schools, these sessions afford opportunities otherwise out of their reach and insure to large numbers of teachers a growth in effectiveness and an enlarging interest in their work which they could obtain in no other way. Of these general advantages, the teachers of mathematics probably enjoy a larger share than any others, since the data seem to indicate that a relatively larger number of teachers are registered for mathematics courses than is the case in other departments.

Finally, one other phase of summer school work should be mentioned, namely, that which has been held at Chatauqua, New York, since 1878. While this began, and was long continued, largely in the form of popular lectures, in more recent years organized class work has been carried on in many departments of the school and college curriculum, including mathematics. The attendance is large and the interest is great. Much attention is given to arithmetic, algebra, and geometry, and no doubt the influence upon the teaching of these subjects is wide-spread and important.

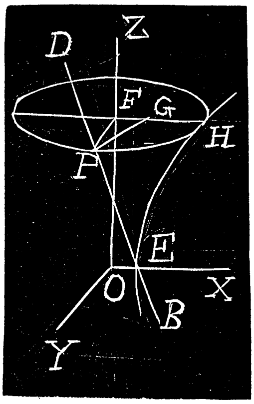
CONCLUSION.

It has been seen that the summer session is a development of the last twenty years, and more extensively of the last ten years. Many colleges and most of the universities are now holding summer sessions. Particular interest centers in those institutions where schools of education have been established on a professional scale or where the department of education is recognized as of leading importance. But even in those institutions where this is not the case, much of the work of the summer session has a pedagogical character due in part to the method of presentation and in part to the critical observation of the instructor by the summer students who are themselves teachers. The normal schools, especially of the middle West, are using the summer sessions to the very great advantage of large numbers of teachers who, otherwise, would be deprived of the opportunity for advancement and growth in the profession of teaching. The department of mathematics has its full share of patronage in the summer sessions and is doing its full quota of work in this form of preparation for teaching.

THE HYPERBOLOID AS A RULED SURFACE.*

By JAMES PERRY WILSON, Columbia University.

If a straight line is revolved around another line not in the same plane, as indicated below, it will generate a hyperboloid of one sheet.



Take the Z-axis as the fixed line. We can then fix the other axes so that the generating line, in its initial position BD, will be parallel to the YZ-plane, and will intersect the X-axis at E. The equations of the generating line in this position will then be of the form $x=a$, $y=mz$.

As the line is revolved around the Z-axis, every point on the line will generate a circle around the Z-axis; and the locus of the intersections of these circles with the XZ-plane will indicate the nature of the surface generated by the line.

Consider any point $P(a, mz, z)$ on the line BD. As the line is revolved around OZ, P will describe a circle around OZ, whose radius $FP=\sqrt{(FG^2+GP^2)}=\sqrt{(a^2+m^2z^2)}$. This circle will intersect the XZ-plane at H; then

$$FH=x=\sqrt{(a^2+m^2z^2)}\ldots(1).$$

Since P is any point on the generating line, in its initial position, equation (1) represents the locus of the intersections with the XZ-plane, of the circles generated by all the points on the moving line.

From (1), by rationalizing, transposing, and dividing by a^2, we obtain

$$\frac{x^2}{a^2}-\frac{m^2z^2}{a^2}=1.$$

Putting $\frac{a^2}{m^2}=c^2$, we obtain

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1\ldots(2),$$

which is the standard equation of the hyperbola; and the surface generated

*This note was presented by a student in a beginner's class in solid analytical geometry. Its extreme simplicity seems to warrant its publication.

by revolving BD around OZ is the same as that generated by revolving the hyperbola (2) around OZ, its conjugate axis; that is, the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=1.$$

If the moving line is perpendicular to the fixed line, it will lie wholly in the plane $Z=k$, and the hyperboloid will degenerate into the plane figure $Z=k$, $x^2+y^2 \geqq a^2$, a in this case being the shortest distance between the lines OZ and BD.

THE SOLUTION OF AN EQUATION BY A FRAME.

By T. M. BLAKSLEE, Ames, Iowa.

The *frame* of the equation

$$f(x)=x^n+bx^{n-1}+cx^{n-2}+dx^{n-3}+\ldots=0,$$

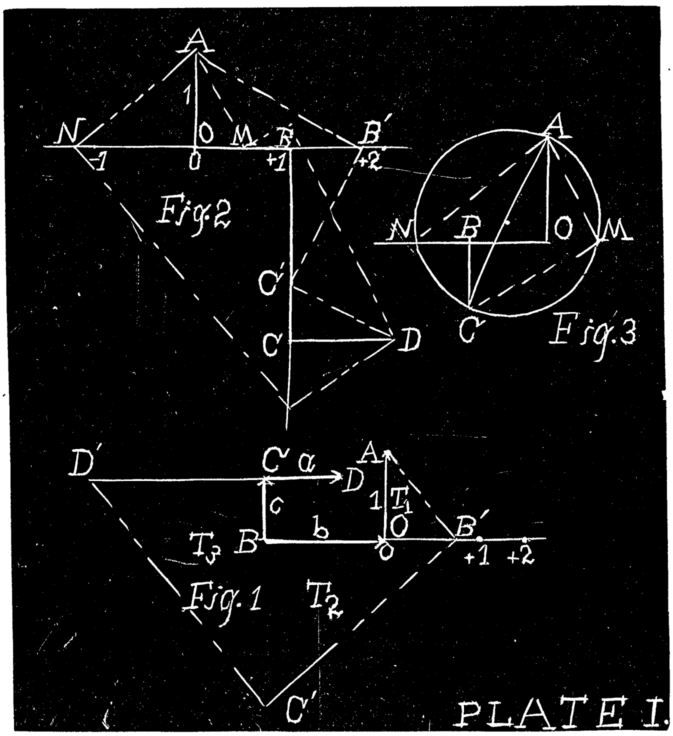


PLATE I. $OB'=r$, $C'B=r^2+br$, $D'C=r^2+br^2+c$ in Fig. 1. Also, $b'=r+b=BB'$, $c'=r^2+br+c=C'C$. In figure after 1, b, c, ... are omitted as their use is evident from Fig. 1.

(see Fig. 1) consists of a succession of strokes, AO, OB, BC, CD, ..., successive strokes being at right angles, $AO=a=-1$, $OB=-b$, $BC=c$, $CD=d$, $DE=-e$, $EF=-f$, $FG=-g$, ...

To substitute r for x, OB' representing r, draw the *path* AB', $B'C'$, $C'D'$, ..., T_1, T_2, T_3, ... These are easily seen to be similar, and as OB' is r times OA, BC is r times BB', ... Hence the values on the figure. If $f(x)=x^3+bx^2+cx+d$, $f(r)=D'D$. If r is a root of $f(x)=0$, D' falls at D.

To draw the path for $x=r$, place one edge of a rectangular card through A and a point near B', place another card against it and a third against the second and so the proper edge passes through D. Now A and D remaining fixed, slide the cards to locate B' and C'. A little practice makes this easy. Fig. 2 solves $x^3-x^2-2x+1=0$. Horners method will give as many figures as wished. After locating one root, *e. g.*, that near $+2$, we may draw, Fig. 3, the frame of the quadratic having the other two roots. In this b and c are the b' and c' of Fig. 2. Just so in the numerical way,

$$
\begin{array}{lc}
 & \begin{array}{ccccccc|l} 1 & + & b & + & c & + & d & \underline{r} \\ & & r & & r^2+br & & & \end{array} \\
\cline{2-2}
x^2+b'x+c'=0 & 1,\ r+b=b',\ r^2+br+c=c',\ d'=0
\end{array}
$$

As the Runge Circle cuts the ray of b the other two roots are real. They should check with those of Fig. 2.

As a second illustration we will take the cubic $x^3-4x^2+9x-10=0$. Fig. 4 gives one real root. As the Runge Circle (*i. e.*, the circle on the diameter AC, Fig. 5) does not cut the ray of b, the other two roots are complex.

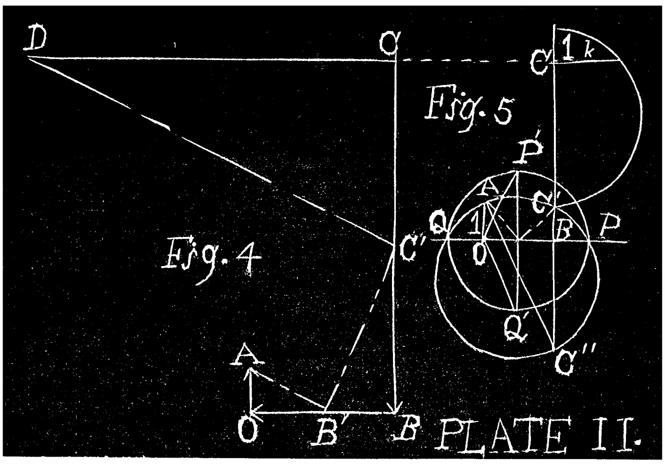


PLATE II. As $OB=b$ (numerically)$=2h$, it is evident $BC'=h^2$.

Of the four methods that I have devised for the solution of a quadratic with complex roots I give two. Of these, II seems the "slicker." To consider all of them in their relations would alone be enough for a paper.

Starting with the frame *AOBC* (Fig. 5), b and c being the b' and c' of Fig. 4, we draw the path $AB'C'$ for *equal roots*. If the roots are $h+ki$ and $h-ki$, $h=-\frac{1}{2}b$, $h^2+k^2=c$. Therefore, $C'C=c-h^2=k^2$.

I. Find k as in upper part of Fig. 5. II. Make $C'C''=C'C$, and solve the frame $AOBC''$ giving the root-points P and Q. Then P' and Q' are the root-points of the given quadratic frame $AOBC$, since $y^2-2hy=k^2-h^2$ gives $y=h\pm k$.

The *linkage* of $f(x)$ for a given complex value of x is the succession of strokes, or links, which starting from the origin has the terminus of one link as the origin of the next, and each link represents the corresponding term of $f(x)$. If the value of x used is a root of $f(x)$ the last link terminates at the origin. The point representing x, or "point x," is the root-point. The point X representing $f(x)$, or "point X," is the function-point. As we may solve $x^n=N$ by a frame, *e. g.*, $x^3+0x^2+0x-N=0$, Fig. 6 gives an easy way of finding the lengths of x^2, x^3, x^4, ... Their slants being two, three, four, ... times that of x. Fig. 7 gives the linkages for $x^3-4x^2+9x-10=0$

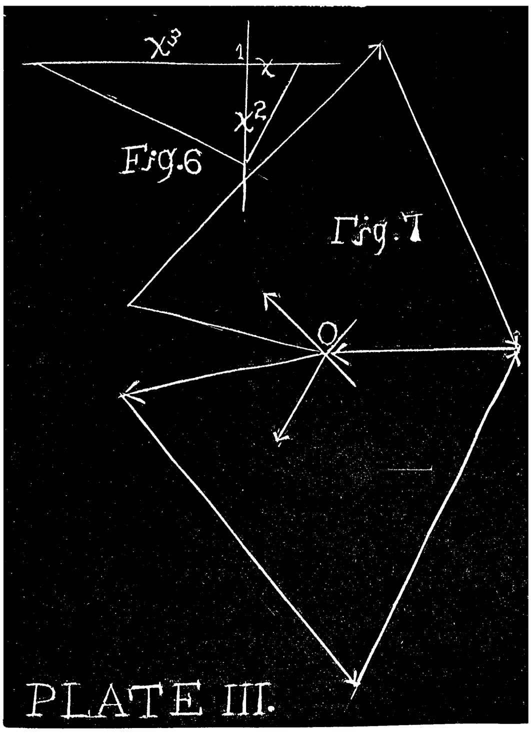


PLATE III. The successive strokes in Fig. 7 from O to O again are the successive terms of the polynomial $x^3-4x^2+9x-10$. Those in the upper part of the figure (light strokes) are for $x=x_2$. Those in the lower part of the figure (heavy strokes) are for $x=x_1$.

for $x_1=OP$, $x_2=OQ$, the unit being two-fifths that of Fig. 5. It is easy to show that the termini of corresponding links are conjugate points symmetrical as to the axis of reals. Hence if X_1 falls at the origin, so does X_2. We easily have a graphical proof of the theorem. Complex roots enter $f(x)=0$ in conjugate pairs.

Let us now consider a more general method in which all roots are regarded as of the form $h+ki$, where either h or k may be zero.

The chief interest in the method is its generality. The "double position" method used is: Having found R_1 and R_2, the function-points corresponding to r_1 and r_2, construct on r_1r_2 a triangle r_1r_20 similar to R_1R_2O, and take 0 as a first approximation to the root-point of the origin O. In Figs. 8, 9, and 10, the method is applied to $x^4-5x^3+13x^2-19x+10=0$.

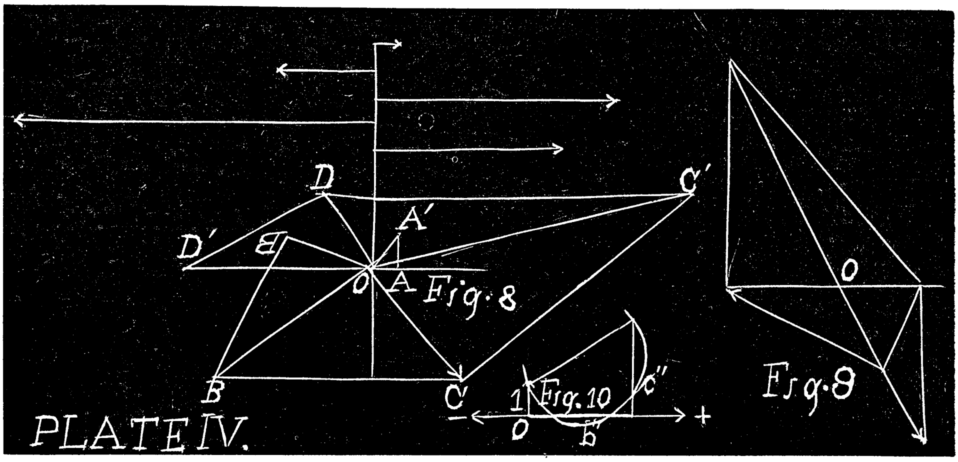


PLATE IV. Referring to Fig. 1 and the note concerning it, it is seen that, if r be for r_1, $OA'=r$, $A'B=b$. $\therefore$ $OB=b'$, $OB'=b'r$, $B'C=c$. $\therefore$ $OC=c'$, $OC'=c'r$, $C'D=d$. $\therefore$ $OD=d'$ and $OD'=d'r$. As $OD'+e=0$. Thus is the last stroke terminate at O, r is a root.

It is to be remembered that, "Geometric multiplication consists in doing with the multiplicand (as regards slant and slide, *i. e.*, turning and stretching) what must be done to the initial unit, (1_0), to obtain the multiplier;" also that, "The geometric sum of two consecutive strokes is the stroke from the first origin to the last terminus." In Fig. 8, *for the multiplication* the triangles OAA', OBB', OCC' and ODD' are similar. *For the addition*, $OB=OA'+A'B$ or $r+b=b'$, $OC=OB'+B'C=b'r+c=r^2+br+c=c'-\ldots$ In (9) a', b', c', d' of (8) are used to find a'', b'', c''. These are all reals. Thus in (10) we have the ordinary frame for the quadratic with roots 1 and 2. (Unit in (9) and (10) twice that in (8).) Thus the roots of $x^4-5x^3+13x^2-19x+10=0$ are $1+2i$, $1-2i$, $+1$, and $+2$.

It seems to me that, if we carry out what is touched upon in connection with Fig. 5, it will be seen that it is more logical to regard the roots as $1\pm 2i$, where $k=2$ and $\frac{3}{2}\pm(\frac{1}{2}i)i$ where $k=\frac{1}{2}i$ than to say they are $1+2i$, $1-2i$, $1+0.i$, and $2+0.i$.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

356 Proposed by ARTEMAS MARTIN, Ph. D., Washington, D. C.

Solve by quadratics, if possible, the equations,

$$w(x+y+z)=a, \quad x(w+y+z)=b,$$
$$y(w+x+z)=c, \quad z(w+x+y)=d.$$

[From the *Mathematical Magazine*, Vol. II, p. 256.]

No satisfactory solution of this problem has been received.

357. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Illinois.

Solve the system

$$\sqrt{(x^2+a^2+b^2+c^2)}=\sqrt{(y^2+b^2+c^2)}+\sqrt{(z^2+b^2+c^2)},$$
$$\sqrt{(y^2+a^2+b^2+c^2)}=\sqrt{(x^2+c^2+a^2)}+\sqrt{(z^2+c^2+a^2)},$$
$$\sqrt{(z^2+a^2+b^2+c^2)}=\sqrt{(x^2+a^2+b^2)}+\sqrt{(y^2+a^2+b^2)}.$$

Solution by ARTEMAS MARTIN, LL.D., Editor and Publisher, Mathematical Magazine, Washington, D. C.

Let $u=x^2+a^2+b^2+c^2$, $v=y^2+a^2+b^2+c^2$, $w=z^2+a^2+b^2+c^2$, then

$$x^2+c^2+a^2=u-b^2, \quad x^2+a^2+b^2=u-c^2;$$
$$y^2+b^2+c^2=v-a^2, \quad y^2+a^2+b^2=v-c^2;$$
$$z^2+b^2+c^2=w-a^2, \quad z^2+c^2+a^2=w-b^2;$$

and $x=\sqrt{(u-a^2-b^2-c^2)}$, $y=\sqrt{(v-a^2-b^2-c^2)}$, $z=\sqrt{(w-a^2-b^2-c^2)}$.

Substituting in the given equations,

$$\sqrt{u}=\sqrt{(v-a^2)}+\sqrt{(w-a^2)}\ldots(4),$$
$$\sqrt{v}=\sqrt{(u-b^2)}+\sqrt{(w-b^2)}\ldots(5),$$
$$\sqrt{w}=\sqrt{(u-c^2)}+\sqrt{(v-c^2)}\ldots(6).$$

Squaring (4), (5), and (6), transposing, etc.,

$$2\sqrt{(v-a^2)}\sqrt{(w-a^2)}=u-v-w+2a^2\ldots(7),$$
$$2\sqrt{(u-b^2)}\sqrt{(w-b^2)}=v-u-w+2b^2\ldots(8),$$
$$2\sqrt{(u-c^2)}\sqrt{(v-c^2)}=w-u-v+2c^2\ldots(9).$$

Squaring (7), (8), and (9), transposing, etc.,

$$4vw=(u-v-w)^2+4a^2u\ldots(10),$$
$$4uw=(v-u-w)^2+4b^2v\ldots(11),$$
$$4uv=(w-u-v)^2+4c^2w\ldots(12).$$

Expanding in (10), (11), and (12), transposing, etc.,

$$2uv+2uw+2vw=u^2+v^2+w^2+4a^2u \ldots (13),$$
$$2uv+2uw+2vw=u^2+v^2+w^2+4b^2v \ldots (14),$$
$$2uv+2uw+2vw=u^2+v^2+w^2+4c^2w \ldots (15).$$

From (13), (14), and (15), we have, obviously,

$$4a^2u=4b^2v=4c^2w \ldots (16),$$

which gives $v=\frac{a^2u}{b^2}$ and $w=\frac{a^2u}{c^2}$. Substituting these values of v and w in (13), we get

$$u=\frac{4a^2b^4c^4}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)}.$$

Since $v=\frac{a^2u}{b^2}$ and $w=\frac{a^2u}{c^2}$, we have

$$v=\frac{4b^2a^4c^4}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)},$$

$$w=\frac{4c^2a^4b^4}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)}.$$

Substituting the above values of u, v, w in the expressions for x, y, z, we have, finally,

$$x=\sqrt{\left(\frac{4a^2b^4c^4-2a^2b^2c^2(a^2+b^2+c^2)^2+(a^2+b^2+c^2)(a^4b^4+a^4c^4+b^4c^4)}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)}\right)},$$

$$y=\sqrt{\left(\frac{4b^2a^4c^4-2a^2b^2c^2(a^2+b^2+c^2)^2+(a^2+b^2+c^2)(a^4b^4+a^4c^4+b^4c^4)}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)}\right)},$$

$$z=\sqrt{\left(\frac{4c^2a^4b^4-2a^2b^2c^2(a^2+b^2+c^2)^2+(a^2+b^2+c^2)(a^4b^4+a^4c^4+b^4c^4)}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)}\right)}.$$

GEOMETRY.

378. **Proposed by G. I. HOPKINS, A. M., Instructor in Mathematics and Astronomy, Manchester High School Manchester, N. H.**

In the triangle AED, the lines BE and CE are drawn to the points B and C in the base of the triangle. If $AE=100$, $ED=125$, $BC=60$ and $\angle AEC=\angle BED=$a right angle, compute AB, BE, EC, and CD.

II. **Solution by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.**

The solution by A. H. Holmes is correct in principle, but he has made some numerical mistake. The method of successive approximation is not only easier, but has the advantage that the work checks itself (except the last operation).

Using Mr. Holmes' notation, we have,

$$4\sin\theta=5\sin\psi\ldots(1);$$
$$25\left(\frac{1}{\cos\psi}-\cos\psi\right)+20\left(\frac{1}{\cos\theta}-\cos\theta\right)=12\ldots(2).$$

Let $\theta=35°$. Then from (1), $\sin\theta=.4589$ and $\psi=27°\ 19'$, $\cos\psi=.8885$. In (2), the first member has the value 13.955.

Let $\theta=33°$. From (1), $\psi=25°\ 50'$. First member of (2) $=12.3445$.

Let $\theta=32°\ 34'$, $\psi=25°\ 30'\ 20''$. First member of (2) $=12.0065$.

Let $\theta=32°\ 33'$, $\psi=25°\ 29'\ 41''$. First member of (2) $=11.999915$.

The work can be shortened by using interpolation just as is done in using logarithm tables.

With these last values of θ and ψ, we get,

$$AB=58.635,\quad BE=59.608,\quad EC=63.830,\quad CD=78.485.$$

NOTES AND NEWS.

Dr. E. B. Stouffer, who received his Ph.D. degree from the University of Illinois in June, has been appointed instructor in mathematics in the same institution. M.

Miss Josephine Burns, who was a graduate student in the University of Illinois during the past year, has accepted a fellowship in mathematics in the University of Wisconsin for the coming year. M.

Dr. Thomas Buck and Dr. L. I. Neikirk resigned their instructorships in the University of Illinois to accept similar positions in the University of California and the University of Washington, respectively. M.

At the University of Illinois, Professor H. L. Rietz was promoted from an assisstant professorship to an associate professorship in mathematics. Professor Rietz gives courses on the theory of statistics and the mathematics of investment as well as courses in pure mathematics. M.

The fifth International Congress of Mathematicians will be held in Cambridge, England, from August 22 to August 28, 1912. It will be organized by the Cambridge Philosophical Society with the coöperation of the London Mathematical Society. This will be the first international congress of mathematicians to be held in an English-speaking country, and it is expected that a large number of Americans will attend. The preceding four congresses were held as follows: Zurich, Switzerland, August, 1897; Paris, France, August, 1900; Heidelberg, Germany, August, 1904; Rome, Italy, April, 1908. M.

B. G. Teubner of Leipzig announces that they have in preparation five large volumes on the teaching of mathematics in Germany. A number of parts have already appeared and can be purchased separately. The general titles of the five volumes are as follows: The higher institutions of learning in northern Germany, The higher institutions of learning in middle and southern Germany, Special questions relating to higher mathematical education, Mathematics in the technical schools, and Mathematics in the public schools and in the normal schools. Felix Klein of Göttingen is general editor of the series. M.

On April 5th of the present year the Spanish Mathematical Society was organized in Madrid, Spain. During the following month this Society began the publication of a monthly periodicel called *Revista de la Sociedad Matematica Espanola.* The first two numbers of this periodical contain 40 and 36 pages, respectively, and the subject matter of these numbers inspires considerable confidence in the success of the journal. It is divided into seven sections bearing the following headings: Biography, articles, bibliography, news, vocabulary, intermedium, and problems. American mathematicians should take an especial interest in this new Society and its journal in view of the extensive use of the Spanish language on this continent and in our recently acquired possessions. The rapid recent advances of Italy towards mathematical eminence may inspire us with the hope that the Spanish Mathematical Society may be the beginning of a new mathematical era among those who use the Spanish language. M.

Two matters of interest to mathematical readers received special attention at the meeting of the National Education Association in San Francisco in July. One was the presentation of the modified schemes of admission recently promulgated at Harvard University and at the Univerity of Chicago. The Harvard program was presented by Assistant Professor Davis of the department of physics, who gave in exceedingly clear manner a

most illuminating exposition of Harvard's practice heretofore with reference to the admission of students, and of the proposed system which goes into operation for the first time this fall. In a word, the new plan is a well considered combination of certificates from the schools and examinations at the University. It is proposed to accept the statement from the schools as to the ground covered in the various subjects by the candidate and his general character and standing as a student during his preparatory course; and no longer to set examinations for the purpose of determining his *quantitative* attainment in all the varied ramifications of his four years' work. Examinations, however, will still be given, but of a different character from those of the past. They will be set in four different subjects, three of which are specified by the University and one is to be selected by the candidate. These will be given all at one time and only when the candidate is ready for admission. The purpose of the examination is entirely *qualitative,* namely, to ascertain the candidate's ability to think and to express himself along the line of his preparatory study, and with this in view he is allowed to choose his own subject for one of the tests, presumably the one which is his favorite and in which he can best display his real strength. These papers are not to be graded by any percentage system and there will be no *passing* in one or more of the *separate* subjects. The candidate will either be admitted *without conditions* on the strength of his general ability as shown in these examination papers and by his credentials from the school, or he will be rejected and debarred from further opportunity for trial at that time. This plan is without doubt a great improvement over the old one and will aid greatly both in bringing Harvard into closer touch with the public high schools from which source she has hitherto drawn comparatively few of her students, and also in confounding the host of mere coaching schools which have long prospered as expert crammers for Harvard's special type of quantitative examinations.

The new Chicago program was supported by Associate Professors O. W. Caldwell of the department of biology in the School of Education, and H. E. Slaught of the department of mathematics, both of whom were members of a large committee of the faculty who have had the matter under consideration for the past two years. As is well known, the University of Chicago admits students to its collegiate departments on certificates from accredited schools. Hence the question was not one of examinations in one form or another set by the University, but it was rather a reconsideration of the relations of the University to the schools and the development of a better scheme of *coöporations.* To this end a committee of five representatives of the University and five representatives of the secondary schools in the Middle West have been holding numerous all day sessions during the past year and the results of their work were continually available for the committee of the faculty above mentioned. The chief considerations embodied in the new programs adopted by the faculty last June may be sum-

marized as follows, and it will be seen that these points refer not only to the work of the preparatory school but also to that of the University as related to the schools:

(1) *Itemized* specifications of the preparatory work are to be replaced by *group* specifications which will allow a wide range of selection but will demand *continuity* of work along two or three lines chosen by the pupil. Of the fifteen units required for admission a student from an accredited four years high school must present three units of English (the only specifically named subject), together with a major sequence of three units from one of groups: Ancient languages, modern languages, history, mathematics, science; a minor sequence of two units from another one of these groups; and finally two units chosen at will from these groups, making ten units in all from the standard academic subjects. The remainining five units of the fifteen he may present from any subjects which the accredited school accepts toward its graduation diploma.

(2) The *group continuity* thus demanded in the preparatory school must then be continued in the first year in college, as follows: (a) The student must continue the study of English as one of his freshman subjects. (2) He must also continue his major or minor sequence or in special cases a subject which he studied during his last year in the preparatory course.

(3) *Breadth of view* is assured by requiring the equivalent of two units in each of the groups: Modern languages, historical subjects (including economics and sociology), mathematics, sciences, either in the preparatory school or during the first two years in college.

(4) *Depth of attainment* is assured by requiring during the four years in college one sequence of nine majors in some chosen department or in related departments, and one sequence of six majors in some other department. (A major is a course four or five hours per week for twelve weeks.)

(5) It is further understood that the future accrediting of schools is to be based upon a carefully prepared system of comparative records running through both secondary school and college and that the mutual relationship and mutual responsibility of both school and college teachers are to be fostered and developed by a scheme of mutual visitation whereby the secondary teacher is to see how the pupil is handled in college and the college teacher is to come into closer touch with the problem as it presents itself to the schools.

It is believed that the scheme of corporation between the secondary school and the University will greatly aid in bridging the unwarranted and artificial chasm which has heretofore existed between them, will assist the student in his first year in college to avoid many of the fatal pit-falls which have beset him, and will enable the schools to devote themselves to the task of serving their communities most advantageously rather than to the patchwork business of meeting the arbitrary details of minutely specified college entrance requirements, since by this program any student who has success-

fully completed a four years' high school course and has devoted himself within reasonable limitations to *serious and prolonged study* of English and two other academic subjects may be admitted to college without conditions and may there find opportunity to continue his work for a college degree.

The question of special interest to teachers of mathematics is the fact that, according to the Chicago scheme, mathematics is made coordinate with modern an ancient languages, history and the sciences as *one* of five groups from which the two sequences must be chosen for college admission.

The other items of interest to teachers of mathematics is the preliminary report of the National Committee on Geometry which was presented at the meeting of the sections by the chairman, H. E. Slaught, of Chicago. This report is a pamphlet of about 80 pages reprinted from *School Science and Mathematics,* where it appeared in three installments in April, May, and June. The printed copies being in the hands of the audience, the whole time of the session was devoted to an exposition of the salient points by the chairman and to formal and informal discussion and criticism. As was to be expected a very large majority of the nearly two hundred present were teachers from the Pacific Coast, and the warmth and interest manifested in the discussions were characteristic of the free and untrammeled spirit of the West. A full report of the meeting will be printed in the proceedings of the Association, including the historical introduction of Professor Cojari. The remaining portions of the report will be carefully gone over by the committee during the present year in the light of all the suggestions and criticisms received and will be presented as a final report at the meeting of the Association next summer.

It may be said that very few unfavorable criticisms have been received from any source, and these are of minor importance, while the favorable comments are many and strong. The two points most ardently commended are (1) the attempt to distinguish the theorem of geometry on the basis of emphasis and to enable the teacher to get a view point of perspective in the subject, and (2), the attempt to improve the point of view with respect to problems and exercises.

An appropriation was voted by the finance committee of the Association for printing a large edition of the reports as soon as the committees shall be able to put the work into final form, at which time announcement will be made as to the terms on which it may be obtained by all who wish to receive it.

Professor A. L. McCarty is now at the head of the Department of Mathematics in the State Normal School, Cape Girardeau, Mo. F.

Professor Alois F. Kovarik has just returned to his work in the Department of Physics in the University of Minnesota, after a two years' leave of absence in research study in radio-activity with Professor Rutherford in England. F.

Professor B. F. Yanney, formerly of Mount Union College, is now at the head of the Department of Mathematics in Wooster University, Wooster, Ohio. F.

BOOKS.

Elements of the Differential and Integral Calculus. By Wm. Anthony Granville, Ph. D., President of Pennsylvania College, with the Editorial Co-operation of Percy F. Smith, Ph. D., Professor of Mathematics in the Sheffield Scientific School of Yale University. Revised Edition. 8vo. Cloth. xv+463 pages. Price $2.50. Boston and Chicago: Ginn & Co.

This book has been very favorably received by teachers of the Calculus. Portraits of the two discoverers of the Calculus are inserted in the present edition. F.

Physical Measurements. By A. Wilmer Duff, Professor of Physics in the Worcester Polytechnic Institute, and Arthur W. Ewell, Professor of Physics, in the Worcester Polytechnic Institute. Second Edition, Revised and Enlarged, with 78 Illustrations. 8vo. Cloth. x+258 pages. Price, $1.50. Philadelphia: P. Blakeston's Son & Co.

This manual contains a large variety of the most important experiments in the various subjects of Physics, such as Mechanics, Heat, Sound, Light, and Electricity and Magnetism. The experiments can be made with a minimum amount of apparatus. F.

Exercises From Algebra for Secondary Schools. By Charless Davison, Sc. D., Mathematical Master of King Edward's High School, Birmingham, Eng. 8vo. Cloth, vi+320 pp. Price with answers, $1.00. Cambridge, England: The University Press. G. P. Putnam's Sons, American Agents.

The book contains a fine collection of exercises and problems in the various subjects in elementary algebra from addition through the progressions, logarithms, permutation and combination, convergency and divergency of series, and exponential and logarithmic series. The answers are given at the end of the volume. The book will be found very useful to those teachers who wish to supplement the problems of the texts with well chosen problems from outside sources. F.

Principles of Physics, Designed for the Use as a Text Book of General Physics, by William Francis Magie, Ph. D., Henry Professor of Physics in Princeton University. 8vo. Cloth. ix+570 pages. Price, $3.00. New York: The Century Co.

This is a text book of rare excellence and one that will, we predict, be extensively used in our best colleges and universities. The book develops the subject in the historical order, thus putting in the clearest light those facts which are fundamental, showing the way theory grows out of facts, and leading the mind of the student along the pathway of discovery. The facts first discovered were those lying on the surface and thus most exposed. These are generally the easiest to understand. This is the reason why the historical order of study is pedagogically the best. Only a minimum of mathematics is required in reading this text. F.

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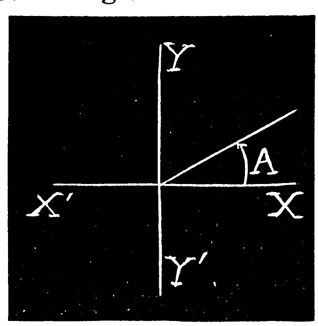
REDUCTION OF THE TRIGONOMETRIC FUNCTIONS OF ANY ANGLE TO THE FUNCTIONS OF THE ANGLES IN A SMALL INTERVAL.

By DR. G. A. MILLER.

In view of the great practical importance of the reduction of the trigonometric functions of an angle to the similar functions of an angle in an interval which does not exceed 90°, it may be of interest to present a method of reduction which is simple and has some points of novelty even if this method is not decidedly superior to the one commonly employed in the text-books on trigonometry. The method has, however, the strong point in its favor that it is applicable to many other problems and exhibits deep contact which does not appear so clearly in the methods commonly employed.

For simplicity of statement we shall always assume that the angle A under consideration has its vertex at the origin of a rectangular system of coördinates and that the initial line coincides with the positive part of the x-axis. It is easy to see that we may find the geometric* angle which corresponds to the complement of A by reflecting the terminal line of A on the bisector of the first and third quadrants.



Similarly, we may find the supplement of A by reflecting the terminal line of A on the y-axis.

If we perform the operation of taking the complement (c) and of taking the supplement (s) of the finite angle A and of the resulting angles, we obtain, in general, eight different geometric angles. We never obtain more than eight such angles, and we always obtain exactly eight except when the terminal line of A lies either on one of the coördinate axes or on a bisector of one of the

* It is well known that each geometric angle has an infinite number of different measures. These measures will be called the analytic angles corresponding to the given geometric angle. Hence α and $\alpha+360°$ represent different analytic angles but they represent the same geometric angles. The trigonometric functions are based upon the geometric angles.

quadrants. For instance, when $A=20^\circ$ the following eight angles form a complete set of conjugates with respect to the operations c and s:

$$20^\circ,\ 70^\circ,\ 110^\circ,\ 160^\circ,\ 200^\circ,\ 250^\circ,\ 290^\circ,\ 340^\circ.$$

That is, we obtain no new geometric angle by performing the operations c and s upon any one of these eight angles, and if we perform these operations successively upon any one of them and upon the angles which result we obtain *all* of them.

In general, if we perform upon the angle A, and upon the resulting angles, the operations c and s we obtain the following eight angles:

$$A,\ 90^\circ-A,\ 90^\circ+A,\ 180^\circ+A,\ 270^\circ-A,\ 270^\circ+A,\ 180^\circ-A.$$

It may be observed that these eight angles are similarly situated as regards the quadrants, and that each one of the octants into which the plane is divided by the coördinate axes and the bisectors of the quadrants contains one and only one of these angles if A is not on the boundary of an octant. If A is on the boundary of such an octant the given eight angles reduce to four distinct geometric angles which include either the beginning line or the terminal line of every octant. This proves the following fundamental theorem:

By performing successively upon any angle and upon the resulting angles the operations of getting its complement and its supplement we always obtain a set of angles which include one and only one geometric angle in the interval from 0° to 45°, and all of the angles of this set can be obtained from each one of them by means of these operations.

From the preceding theorem it results directly that if we know all the trigonometric functions of the angles in the interval from 0° to 45°, and if we also know how to find these functions of the complement and of the supplement of A from the functions of A, we can find the trigonometric functions of any angle whatsoever. The finding of the trigonometric functions of any angle is therefore reduced to three tables, as follows:

I. Table giving the trigonometric functions of angles in the interval from 0° to 45°.

II. $\sin A=\cos(90^\circ-A)$, $\cot A=\tan(90^\circ-A)$,
$\cos A-\sin(90^\circ-A)$, $\sec A=\csc(90^\circ-A)$,
$\tan A=\cot(90^\circ-A)$, $\csc A=\sec(90^\circ-A$.

III. $\sin A=\sin(180^\circ-A)$, $\cot A=-\cot(180^\circ-A)$,
$\cos A=-\cos(180^\circ-A)$, $\sec A=-\sec(180^\circ-A)$,
$\tan A=-\tan(180^\circ-A)$, $\csc A=\csc(180^\circ-A)$.

While these three tables are sufficient to find the trigonometric functions of any angle, the other tables generally given in text-books on trigon-

ometry are desirable as they save time. This presentation of the matter is offered mainly on account of its theoretic interest, and because it connects this elementary process in trigonometry with the extensive subject of group theory. In fact, the operations denoted by c and s generate a group of order 8 known as the octic group or the group of the square. From this fact it results directly that the operations c and s could be replaced by any pair of generators of this octic group. In particular, we could replace Table III by one giving the functions of A in terms of functions of $-A$, without changing tables I and II.

The point of view developed above has other advantages as it directs attention to the fact that we replace Table I by one covering a very much smaller interval provided we replaced the other two tables by suitable tables. In fact, Table I could be diminished indefinitely, but such a reduction would have no practical bearing. To indicate changes along this line, which have only theoretical interest, we may observe that Table I could be replaced by one covering only the interval from $0°$ to $22\frac{1}{2}°$, by assuming Table II and replacing Table III by the following:

$$\sin(45°-x)=\tfrac{1}{2}\sqrt{2}(\cos x-\sin x), \quad \cot(45°-x)=\frac{\cos x+\sin x}{\cos x-\sin x},$$

$$\cos(45°-x)=\tfrac{1}{2}\sqrt{2}(\cos x+\sin x), \quad \sec(45°-x)=\frac{\sqrt{2}}{\cos x+\sin x},$$

$$\tan(45°-x)=\frac{\cos x-\sin x}{\cos x+\sin x}, \quad \csc(45°-x)=\frac{\sqrt{2}}{\cos x-\sin x}.$$

In fact, the operations of getting the complement and of subtracting from $45°$ generate a group of order 16, as may readily be verified,* and by these operations an angle is, in general, transformed into one and into only one angle in each of the sixteen equal divisions of the circumangle, starting from $0°$. Hence we can find the trigonometric functions of any angle if we know the functions of all the angles in one of those intervals of $22\frac{1}{2}°$ and the operations expressed in the table ending the preceding paragraph, together with those of Table II. There is evidently no limit to similar developments along these lines.

One of the many general methods which includes the special cases given above and illustrates how the interval of the angles whose trigonometric functions are supposed to be known may be reduced indefinitely, is as follows: The group of movements of the regular polygon of n sides is the dihedral group of order $2n$. This group transforms *transitively* each of the

* *Annals of Mathematics*, Vol. 8 (1907), p. 97.

$2n$ half sides of the regular polygon; that is, it may be represented as a regular substitution group on letters so selected that each letter corresponds to one of these half sides. Hence *the group of movements of the regular polygon contains one and only one operator which transforms a particular half side of the polygon into a given half side.* As this group of movements is dihedral it may be generated by two of its operators of order two.

It is known that subtraction may be regarded geometrically as a reflection of the point representing the subtrahend on the point midway between the minuend and the origin.* If we inscribe in a circle a regular polygon of $2m$ sides, $m>1$, the group of movements of this polygon is clearly generated by a rotation through π on the line of symmetry bisecting the first side and a rotation through π on the y-axis. These rotations are equivalent, respectively, to subtraction from π/m and subtraction from π. If we make $m=2$ we have the group, considered above, formed by the operations of getting the complement and the supplement.

In the second example considered above, $m=4$, and, instead of subtracting from π we subtracted from $\frac{1}{2}\pi$, which is evidently always permissible when m is even and greater than 2. As the table of the trigonometric functions may be confined to the interval $\pi/2m$ provided we can deduce the functions of $\pi-A$, and of $\pi/m-A$ from those of A, this indicates how the necessary Table I may be reduced theoretically to an interval which is less than any given interval. Practically, this method should probably not be employed beyond $m=2$.

* *Annals of Mathematics*, Vol. 6 (1905), p. 41.

A SET OF INDEPENDENT ASSUMPTIONS FOR PROJECTIVE GEOMETRY.*

By N. J. LENNES, Columbia University.

INTRODUCTION.

The purpose of the present paper is sufficiently indicated in the title. Incidentally an elementary statement is made of well-understood principles in the foundations of mathematics, and the actual working of these principles is exhibited in treating the set of assumptions for projective geometry. Other sets of assumptions for projective geometry have been given by Veblen and Young† and by the present writer.‡ It is believed that the set of assumptions given in the present paper will lead to a development of Pro-

* The substance of this paper was presented to the American Mathematical Society in February, 1911.

† Veblen and Young, *Projective Geometry*, Vol. I.

‡ Lennes, "Duality in Projective Geometry," *Annals of Mathematics*, Vol. 11 (1911).

jective Geometry which shall be simpler and at the same time form a more natural sequel to Euclidean Geometry than the developments resulting from the other sets just mentioned.

It is usual to begin any logical discussion such as the present one with a set of definitions of the technical terms to be used. It is obvious, however, that any definition of such a term can only be made in terms of other technical terms which are equally in need of definition. The definitions of these terms will introduce still other terms which in turn require definition, and so on. Hence the process of definition never terminates. One alternative would be defining in a circle which is not permitted in a logical science. The other alternative is the one in use among mathematicians of the present day, namely, starting with a set of undefined terms* which are usually specified at the outset. In this paper the undefined terms are "point," "line," "plane," and an undefined relation "on."

When certain terms have been designated as undefined other terms may be defined in terms of these. Thus, while point, line, and plane are undefined, a *flat pencil of lines* is defined as "the set of all lines in a plane passing through one point of that plane." The objection may now be urged that this gives a definition of flat pencil in terms of undefined words and hence is no definition at all. The answer is that it is the only kind of definition possible.

However the undefined terms are given a sort of indirect definition by the fundamental assumptions in which they occur.† Indeed the undefined terms have no logical content whatever except as it is given them by the assumptions in which they occur.

Clearly assumptions‡ are necessary in any purely deductive system, for a proof in such a system consists simply in showing that according to certain admitted rules of logic a given proposition is a necessary consequence of certain other propositions. Hence it is as impossible to prove every proposition as it is to define every term.

The set of assumptions for a mathematical science must fulfill certain well established requirements. That is, they must be sufficient, independent and consistent. The most important of these is the first, namely, that all the

* It should be clearly understood that there is no such thing as an *undefinable* term. The fact is simply that not *all* terms in any particular logical science can be defined. However, two different writers developing the same subject may use entirely different sets of undefined terms.

† The classical name for what in this paper is called assumption was axiom, which was defined as a self-evident truth. It is obvious, however, that no proposition dealing with abstract, undefined symbols can possibly be self-evident. To avoid the connotation of the old term axiom many recent writers have used other terms such as postulate or assumption. The old Greek distinction between postulate and axiom has entirely disappeared from mathematics (except in some elementary texts whose authors are evidently not acquainted in a vital sense with the modern point of view). This distinction seems to have its basis in a misunderstanding of the nature of unproved propositions in mathematics.

‡ The French term for assumption, which translated reads "unproved proposition," seems the most fortunate of all the names that have been used. An axiom, assumption, postulate, or whatever we call it, is distinguished from the other propositions of the system in which they occur solely by the fact they are accepted arbitrarily and without proof. As a matter of fact there are modern sets of "assumptions" which contain propositions much less obvious to the intuition than are some of the theorems derived from them.

remaining propositions of the science must follow from the assumptions by purely *formal* deduction. This means that at no stage in the argumentation is the intuition or any consideration whatever aside from the propositions used in making the proof permitted to validate a statement or an argument.

The assumptions must thus contain by implication every theorem in the mathematical science in which they occur. This requirement is by no means satisfied by the usual axioms of Euclidean geometry. Thus it is not possible to prove by means of these axioms that the diagonals of a parallelogram meet in a point.

The second requirement, namely, that the assumptions shall be independent, means that no one of them shall be a logical consequence of all the others. The set given in this paper are proved independent in § 4. A proof that one proposition cannot be proved from a given set of propositions is of comparatively recent origin. The well known proof of Lobachevsky (1829) that the assumption of parallels is not a logical consequence of the remaining assumptions of Euclidean geometry is apparently the first one of this kind. Later completely independent systems of assumptions have been given by Hilbert, Veblen, Huntington, and others. The method of proving the independence of a given set of assumptions is discussed in § 4.

A set of assumptions is consistent if no two contradictory propositions can be deduced from them. The method of proving a set of assumptions consistent is discussed in § 3.

A set of assumptions may or may not possess the further quality of being categorical. This property of a set of assumptions is somewhat more difficult to understand than those just described. The aseumptions of the present paper are not categorical. It is shown in § 3 that all these assumptions are satisfied by certain finite sets of points, lines and planes. Obviously they are also satisfied by the ordinary projective geometry. That is, these assumptions are satisfied by two essentially different systems. If, on the other hand, a set of assumptions is such shat any two systems which satisfy all its assumptions are essentially alike, the set is said to be categorical.*

In the present paper it will turn out that the undefined terms point, line, and plane are used in the sense ascribed to them in ordinary projective geometry. A point is "on" a line or plane if it lies in the line or plane in the ordinary sense. In this case the line and plane are also said to be "on" the point. A line is "on" a plane if it lies in the plane, and in this case the plane is also said to be "on" the line. The statements just given are by no means definitions in the logical sense, but merely informal descriptive statements to aid the intuition in giving concrete form to the abstract symbols and propositions that follow. It should be noted that such concrete form is no part of the logical proceedure but simply an aid to the memory in retaining the various steps.

* For a more precise statement of this matter see E. V. Huntington, *Annals of Mathematics*, Vol. 6 (1906), pp. 1-43.

One of the most far-reaching theorems of Projective Geometry is the so-called Principle of Duality, namely, *Any valid proposition stated in terms of "point," "line," and "plane" and the relation "on" is valid if point and plane are interchanged.*

In the older treatments of projective geometry this general proposition was not proved, the writers contenting themselves with writing the dual theorems and their proofs side by side, thus exhibiting the principle as each theorem arose. In the treatment of Veblen and Young noted above the theorem is proved in its entirety but only after a considerably difficult and complicated argumentation. The undefined terms of their system are "point," and "line," and the relation "on." "Plane" is then defined as a class of points by means of certain collinearities. Since point is an undefined term, while plane is defined, a completely dual treatment from the outset is impossible.

In the present paper the fundamental propositions form a dual set. That is, they are not changed if the words point and plane are interchanged throughout. Hence it follows without further argument that the set of all theorems provable from these them must be dual. That is, for every such theorem containing point and plane there is another exactly like it except that the words point and plane have changed places.

§ 1. Fundamental Propositions.

I. *The relation "on" is a reciprocal relation.*

This proposition may be stated more in detail as follows:

1. "The point A is on the line l" is equivalent to "the line l is on the point A."

2. "The point A is on the plane α" is equivalent to "the plane α is on the point A."

3. "The line l is on the plane α" is equivalent to "the plane α is on the line l."

Each of these propositions may be split into two. Thus I may be written:

(a) "The point A is on the line l" implies "the line l is on the point A."

(b) "The line l is on the point A" implies "the point A is on the line l."

The three propositions (1), (2), (3) will be referred to as I_1, I_2, I_3.

II. *If the point A and the plane α are each on a line l then A is on α.*

III_1. *Any two points are on at least one line.*

III_2. *Any two planes are on at least one line.*

IV_1. *Two points are on not more than one line.*

IV_2. *Two planes are on not more than one line.*

Definitions. Points which are on the same line are *collinear*. Otherwise they are *noncollinear*. Planes which are on the same line are *collinear*, and otherwise they are *noncollinear*.

V_1. *Three points are on at least one plane.*

V_2. *Three planes are on at least one point.*

VI_1. *Three noncollinear points are on not more than one plane.*

VI_2. *Three noncollinear planes are on not more than one point.*

VII. *The class of elements known as lines consists of at least one element.*

$VIII_1$. *On every line there are at least three points.*

$VIII_2$. *On every line there are at least three planes.*

IX_1. *On every plane there are at least three noncollinear points.*

IX_2. *On every point there are at least three noncollinear planes.*

Propositions I, ..., IX_2 are obviously dual with respect to point and plane.

§ 2. Assumptions and Theorems.

$I_{1,2,3}$; II, III_2, V_1, $V_{1,2}$; VI_1, VII; $VIII_{1,2}$; IX_1 form a set of independent assumptions and all theorems of this section are based on them. III_1; IV_2; VI_2; IX_2 are theorems. For proofs of the independence of the postulates see § 4. III_1, IV_2, VI_2 and IX_2 are proved in this section.

IV_2, *Two planes are on not more than one line.*

Proof. Given the distinct planes α and β which we suppose to be on each of the two lines l_1 and l_2. By $VIII_1$ there are three points on each line, and hence by IV_1 there are among these points a set of three noncollinear points A, B, C. By II each of the points A, B, C is on α and also on β. But by VI_1 A, B, C are on only one plane.

1_1. Theorem. *If a point is on each of two planes then the point is on the line which is on the two planes.*

Proof. Two planes α and β are on one and only one line l (III_2, IV_2); on this line are at least two points A and B. If now there is a point C on both α and β but not on l then A, B, C are noncollinear and hence by VI_1 α and β are the same plane.

VI_2. *Three noncollinear planes are on not more than one point.*

Proof. If there are two points A and B on each of the planes α, β, γ and if α, β are on the line l_1 and β and γ are on the line l_2, then A and B are on l_1 and also on l_2, and hence l_1 and l_2 are identical, and hence α, β, γ are collinear.

2_1. *Not all points on a plane.*

Proof. Let α be any plane. By VII, $VIII_2$ at least a plane β distinct from α. By IX_1, II not all points on β are on α. Hence not all points on α.

III_1. *Any two points are on a line.*

Proof. Let A and B be any two points. By VII a line l exists. If A and B are on l then the theorem is verified. If A and B are not on l there is a point C on l (not collinear with A and B). ($VIII_1$, IV_1). Let D be a point not on the plane α of A, B, C. Then A, B, D are on a plane β. α and β are on one and only one line l_1 (III_2, IV_2). But A and B are on both α and β. Hence l_1 is on A and B (1).

IX_2. *On every point there are at least three noncollinear planes.*

PROOF. Let A be the given point. By VII and $VIII_1$ there exists at least one other point B, and by III_1 there is a line l on A and B. By $VIII_2$ there are three distinct planes α, β, γ on l, and on each plane at least one point not on l. Let C, D, E be such points, one on each of the three planes. Then A, C, D are on a plane Δ. This plane Δ is not on l for in that case it would be on both A and B (II), and hence would be identical with both α and β. But α and β are both on l, whence by (II) any point on both α and β is on l. Therefore Δ is not collinear with α and β. That is, α, β, Δ are three noncollinear planes on A.

Propositions I, ..., IX_2 are now completely established (either by assumption or by proof) and hence we state without further argument the space dual of (1_1).

1_2. THEOREM. *If a plane is on each of two points then the plane is on the line which is on these two points.*

3_1. THEOREM. *If two lines are on a plane they are on a point.*

PROOF. Denote the given plane by α and the given lines on it by l_1 and l_2. By $VIII_2$, IX_1 there is a point C not on α. Let A and B be two points on l_1 and DE two points on l_2. Then ABC are noncollinear as are also CDE (2_1). Denote the planes on these points by β and γ, respectively. By V_2 there is a point F on the planes α, β, γ. Thus F is on l_1 and l_2 (1_2).

3_2. THEOREM. *If two lines are on a point they are on a plane.*

PROOF. This is the dual of (3_1).

Of the usual propositions in projective geometry on the incidence of points, lines, and planes those that now remain are obvious and corollaries of the preceding propositions of this section.

Inasmuch as only a finite number of elements are required to satisfy either set of assumptions (see § 3) some theorems may be trivial, since there may not be a sufficient number of elements present to satisfy the hypothesis. Thus if only the elements required by the assumptions are permitted the Theorem of Desargues is trivial since only seven points are on any one plane.

§ 3. CONSISTENCY OF THE ASSUMPTIONS.

We now inquire whether the given set of assumptions is consistent, independent, and categorical.

That a set of assumptions is consistent is proved by exhibiting a concrete system in which the assumptions are satisfied and which for some reason is regarded as self-consistent. The propositions of § 1 are all satisfied by a system given below in which points are the capital letters, lines the columns each containing three letters, and each set of seven lines forming a rectangle, is a plane.*

* Finite geometries of which the one given here is a type have been studied by a number of writers. Tactical memoranda by E. H. Moore, *American Journal of Mathematics*, Vol. 18 (1896), pp. 264-303, contains a rich collection

(A)

$$
(1)\ \begin{matrix} A & B & C & D & E & F & G \\ B & C & D & E & F & G & A \\ D & E & F & G & A & B & C \end{matrix}
\qquad
(2)\ \begin{matrix} A & B & H & D & I & J & K \\ B & H & D & I & J & K & A \\ D & I & J & K & A & B & H \end{matrix}
\qquad
(3)\ \begin{matrix} A & B & M & D & L & N & O \\ B & M & D & L & N & O & A \\ D & L & N & O & A & B & M \end{matrix}
$$

$$
(4)\ \begin{matrix} B & C & K & E & L & M & J \\ C & K & E & L & M & J & B \\ E & L & M & J & B & C & K \end{matrix}
\qquad
(5)\ \begin{matrix} B & C & N & E & H & I & O \\ C & N & E & H & I & O & B \\ E & H & I & O & B & C & N \end{matrix}
\qquad
(6)\ \begin{matrix} C & D & I & F & K & L & O \\ D & I & F & K & L & O & C \\ F & K & L & O & C & D & I \end{matrix}
$$

$$
(7)\ \begin{matrix} C & D & H & F & J & M & N \\ D & H & F & J & M & N & C \\ F & J & M & N & C & D & H \end{matrix}
\qquad
(8)\ \begin{matrix} D & E & O & G & H & J & L \\ E & O & G & H & J & L & D \\ G & H & J & L & D & E & O \end{matrix}
\qquad
(9)\ \begin{matrix} D & E & M & G & K & I & N \\ E & M & G & K & I & N & D \\ G & K & I & N & D & E & M \end{matrix}
$$

$$
(10)\ \begin{matrix} E & F & L & A & I & N & J \\ F & L & A & I & N & J & E \\ A & I & N & J & E & F & L \end{matrix}
\qquad
(11)\ \begin{matrix} E & F & K & A & O & H & M \\ F & K & A & O & H & M & E \\ A & O & H & M & E & F & K \end{matrix}
\qquad
(12)\ \begin{matrix} F & G & J & B & O & K & N \\ G & J & B & O & K & N & F \\ B & O & K & N & F & G & J \end{matrix}
$$

$$
(13)\ \begin{matrix} F & G & H & B & L & I & M \\ G & H & B & L & I & M & E \\ B & L & I & M & F & G & H \end{matrix}
\qquad
(14)\ \begin{matrix} G & A & I & C & J & O & M \\ A & I & C & J & O & M & C \\ C & J & O & M & G & A & I \end{matrix}
\qquad
(15)\ \begin{matrix} G & A & K & C & H & L & N \\ A & K & C & H & L & N & G \\ C & H & L & N & G & A & K \end{matrix}
$$

In this system there are three planes and also three points on every line. Thus the planes (1), (2), (3) are on the line $\begin{matrix} A \\ B \\ D \end{matrix}$ as are also the points A, B, D. There are seven points on each plane and seven planes on every point. That all the assumptions of § 1 are satisfied by this system is easily verified.

§ 4. Independence of the Assumptions.

We now prove that $\mathrm{I}_{1,2,3}$; II; III_2, IV_1; $\mathrm{V}_{1,2}$; VI_1; VII, $\mathrm{VIII}_{1,2}$; IX_1 of § 1 form a set of independent postulates. $\mathrm{I}_{1,2,3}$; II; III_2, IV_1; $\mathrm{V}_{1,2}$; VI_1 are to be understood as not affirming the *existence* of points, lines and planes. That is, these assumptions would all be satisfied by an entirely vacuous system. For convenience of reference these assumptions are here restated in more explicit form.

---

of such configurations though they are not regarded as finite geometries. Moore first used the matricular notation in this connection. Cf. Veblen and Young, *Projective Geometry*, Vol. 1. In this notation the configuration here given is

$$
\begin{matrix} 15 & 3 & 7 \\ 3 & 35 & 3 \\ 7 & 3 & 15 \end{matrix}
$$

This particular configuration is studied in detail by Moore in a paper on the General Equation of the Seventh and Eighth Degree, *Mathematische Annalen*, Vol. 51 (1899), pp. 417-444. It is also studied by George M. Conwell, *Annals of Mathematics*, Vol. 11 (1910), second series, pp. 60-76. Veblen and Bussey, *Transactions of the American Mathematical Society*, Vol. 7 (1906). pp. 241-259, treated these configurations explicitely as finite geometries and discovered the set of all such geometries.

I_1. *"A point A is on a line l" is equivalent to "the line l is on the point A."*

I_2. *"A point A is on a plane* α*" is equivalent to "the plane* α *is on the point A."*

I_3. *"A line l is on a plane* α*" is equivalent to "the plane* α *is on the line l."*

II. *If a point A and a plane* α *are both on a line l then A is on* α.

III_2. *If planes and lines exist then any two planes are on at least one line.*

IV_1. *If points and lines exist then any two points are on not more than one line.*

V_1. *If points and planes exist then any three points are on at least one plane.*

V_2. *If planes and points exist then any three planes are on at least one point.*

VI_1. *If points and planes exist then three noncollinear points are on not more than one plane.*

VII. *The class whose elements are lines contains at least one element.*

$VIII_1$. *On every line there are at least three points.*

$VIII_2$. *On every line there are at least three planes.*

IX_1. *On every plane there are at least three noncollinear points.*

INDEPENDENCE OF I_1. To prove the independence of I_1 modify the concrete system (A) of § 3 so that the line $\begin{matrix}A\\B\\D\end{matrix}$ shall not be *on* the point A while A is *on* this line. In this system every arsumption except I_1 is verified.

I_2. To prove the independence of I_2 modify the system (A) so that plane (1) shall be *on* the points on which (2) now is, and (2) shall be *on* the points on which (1) now is. The points (1) and (2) remain *on* these planes.

I_3. Modify the concrete system (A) so that the line $\begin{matrix}A\\B\\D\end{matrix}$ is not on the plane (1) but plane (1) is on the line.

II. Modify the system (A) so that the points which are now on plane (1) shall be on plane (2), and those now on plane (2) shall be on plane (1). Likewise plane (1) shall be on the points now on plane (2), and plane (2) on the points now on plane (1). Points and lines, and planes and lines are related as in system (A). Thus the line $\begin{matrix}E\\F\\A\end{matrix}$ remains *on* the plane (1) and the points E, F, A are on this line while they are also *on* the plane (3).

III_2. In system (A) omit the line $\begin{matrix}A\\B\\D\end{matrix}$ but not the points A, B, D.

IV_1. In system (A) add the point G to any line on which it now is not.

V_1. To the system (A) add a point Q which shall be on no line and on no plane.

V_2. To prove the independence of this assumption considsr an ordinary projective geometry with one point removed. Then any three planes which pass through this point are *on* no common point.

VI_1. In the system (A) let the point C be *on* plane (2) (as well as on plane (1)) but not on any line in plane (2).

VII. An entirely vacuous system containing no points, lines, or planes.

$VIII_1$. In the system (A) omit all points and planes.

$VIII_2$. In the system (A) omit all planes.

IX_1. Consider a system consisting of planes (1), (2), (3) of (A) modified as follows: $\begin{matrix}A\\B\\D\end{matrix}$ is a line on each plane and A, B, D are points. The remaining points and lines now in these planes shall be some other distinct entities different from either points or lines. Then we have a system consisting of one line with three points on it, and also three planes. Clearly the points are distinct among themselves, as are also the planes.

Hence we have showed that the set of assumptions of this paper are independent.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

383. Proposed by S. A. COREY, Hiteman, Iowa.

Let $ABCDE$ be a pentagon, plane, or gauche, with sides AB, BC, CD, and DE, of lengths, w, x, y, and z, respectively. Construct four other pentagons, $AB_iC_iD_iE_i$, $AB_{ii}C_{ii}D_{ii}E_{ii}$, $AB_{iii}C_{iii}D_{iii}E_{iii}$, and $AB_{iv}C_{iv}D_{iv}E_{iv}$, having a common vertex at A, and with four consecutive sides in each parallel to the corresponding consecutive sides, AB, BC, CD, and DE, in $ABCDE$. Further, let the lengths of the sides AB_i, B_iC_i, C_iD_i, D_iE_i, AB_{ii}, $B_{ii}C_{ii}$, $C_{ii}D_{ii}$, $D_{ii}E_{ii}$, AB_{iii}, $B_{iii}C_{iii}$, $C_{iii}D_{iii}$, $D_{iii}E_{iii}$, AB_{iv}, $B_{iv}C_{iv}$, $C_{iv}D_{iv}$, $D_{iv}E_{iv}$, be $-wW$, xX, yY, zZ, wX, xW, yZ, $-zY$, wY, yW, zX, $-xZ$, wZ, zW, xY, and $-yX$, respectively; the minus sign indicating the reversal of direction of the corresponding side. Prove that $(W^2+X^2+Y^2+Z^2)(w^2+x^2+y^2+z^2)=AE_i^2+AE_{ii}^2+AE_{iii}^2+AE_{iv}^2$.

Solution by the PROPOSER.

Let **w**, **x**, **y**, and **z** represent the vector sides, AB, BC, CD, and DE, respectively, of the pentagon $ABCDE$, the tensors of corresponding vectors being w, x, y, and z. Then in the other four pentagons will $-W\mathbf{w}$, $X\mathbf{x}$, $Y\mathbf{y}$, $Z\mathbf{z}$, $X\mathbf{w}$, $W\mathbf{x}$, $Z\mathbf{y}$, $-Y\mathbf{z}$, $Y\mathbf{w}$, $W\mathbf{y}$, $X\mathbf{z}$, $-Z\mathbf{x}$, $Z\mathbf{w}$, $W\mathbf{z}$, $Y\mathbf{x}$, and $-Y\mathbf{y}$, represent the vector sides AB_i, B_iC_i, C_iD_i, D_iE_i, AB_{ii}, $B_{ii}C_{ii}$, $C_{ii}D_{ii}$, $D_{ii}E_{ii}$, AB_{iii}, $B_{iii}C_{iii}$, $C_{iii}D_{iii}$, $D_{iii}E_{iii}$, AB_{iv}, $B_{iv}C_{iv}$, $C_{iv}D_{iv}$, and $D_{iv}E_{iv}$, respectively. The remaining or fifth side of each of the four pentagons may be represented as the sum of the other four vector sides, whence

$$AE_i=-W\mathbf{w}+X\mathbf{x}+Y\mathbf{y}+Z\mathbf{z},\quad AE_{ii}=X\mathbf{w}+W\mathbf{x}+Z\mathbf{y}-Y\mathbf{z},$$
$$AE_{iii}=Y\mathbf{w}+W\mathbf{y}+X\mathbf{z}-Z\mathbf{x},\quad AE_{iv}=Z\mathbf{w}+W\mathbf{z}+Y\mathbf{x}-X\mathbf{y}.$$

Substituting the squares of these four vector sums for the squares of the fifth sides which constitute the second member of the given equation, expanding and adding, their sum is found to be identical with the first member, due regard being had to the change of signs. Q. E. D.

The foregoing non-geometric solution is given to show the advantage sometimes resulting from the introduction of vectors in the handling of certain classes of geometric problems. In this case the remarkable simplicity and directness of the solution is due to the fact that the equation given in the problem is a geometric interpretation of the algebraic identity,

$$(W^2+X^2+Y^2+Z^2)(w^2+x^2+y^2+z^2)=(-wW+xX+yY+zZ)^2$$
$$+(wX+xW+yZ-zX)^2+(wY+yW+zX-xZ)^2+(wZ+zW+xY-yX)^2.$$

We may observe that every algebraic identity, in which each term is of the second degree in so far as certain letters are concerned, may be given a geometric interpretation if each of such letters be used to represent a certain vector, and if the scalar but not the vector portion of the product be employed in the interpretation. That this is true follows from the fact that the non-commutative character of vector multiplication does not alter or affect the scalar portion of the product, if each term of such product contains either the product of two separate vectors or the square of some one vector; *i. e.*, if no term in the expanded form is of a degree higher or lower than the second in the letters used to designate vectors.

A solution similar to the foregoing may be employed in problem 377.

384. Proposed by S. LEFSEHETZ, Clark University.

Let ABC be a triangle, O a circle tangent to its three sides, T a variable tangent of O, which cuts the sides BC, CA, AB in a, b, c. Oa', Ob', Oc' the perpendiculars in O to Oa, Ob, Oc, cutting, respectively, T in points a', b', c'. Prove that Aa', Bb', Cc' meet in a point t, and find the locus of t when T varies. Purely geometrical proofs wanted.

No solution of this problem has been received.

385. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Given a triangle ABC, find the radius of a circle touching two of its sides and a line parallel to the third, at a distance $d=u+2r$.

Solution by A. H. HOLMES, Brunswick, Maine.

Let a, b, and c be the sides of the given triangle, c the base. Then h=altitude of the triangle$=\dfrac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)]}}{2c}$, and R=radius of the inscribed circle $=\dfrac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)]}}{2(a+b+c)}$.

Put r=radius of circle touching a and b and a line parallel to c at a distance from c, $2r+u$. Then $h:R=h-(2r+u):r$.

$\therefore r=\dfrac{(h-u)R}{h+2R}$.

Putting for h and R their values in terms of a, b, and c, we have,

$$r=\frac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)^2]}-u}{a+b+3c}.$$

CALCULUS.

306. Proposed by FRANCIS RUST, C. E., Pittsburg, Pa.

Express in elliptic integrals: $A_\theta=\displaystyle\int_0^\theta \frac{dx}{\sqrt{(1-x^4)}}$; $0<\theta<1$.

I **Solution by WALTER D. LAMBERT, A. M., University of Pennsylvania, Philadelphia, Pa.**

Correcting the inequality to read $0<\theta<1$, we find (by using the substitution $x=\cos\phi$, and by calling $\alpha=\cos^{-1}\theta$) that

$$A_\theta=\int_0^\theta\frac{dx}{\sqrt{[(1-x^2)(1+x^2)]}}=-\int_{\frac{1}{2}\pi}^{\alpha}\frac{\sin\phi\, d\phi}{\sqrt{[(1-\cos^2\phi)(1+\cos^2\phi)]}}$$

$$=\int_\alpha^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{[2-\sin^2\phi]}}=\frac{1}{\sqrt{2}}\int_\alpha^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{[1-\frac{1}{2}\sin^2\phi]}}$$

$$=\frac{1}{\sqrt{2}}\left[F\left(\frac{\pi}{2}\right)-F(\alpha)\right]\text{ modulus }\frac{1}{\sqrt{2}}$$

where $F\left(\frac{\pi}{2}\right)$ and $F(\alpha)$ are Legendre's elliptic integrals of the first kind.

As a numerical example take $\theta=\frac{1}{2}$. $\therefore\alpha=\frac{1}{3}\pi$.

$A_\theta=\frac{1}{\sqrt{2}}[1.8451-1.1424]=0.5032$, using the four-place "funktionentafeln" of Jahnke and Emde. As a verification, expand $\frac{1}{\sqrt{(1-x^4)}}$ by the binomial theorem and integrate:

$$A_\theta=\int_0^\theta\frac{dx}{\sqrt{[1-x^4]}}=\int_0^\theta(1+\tfrac{1}{2}x^4+\tfrac{3}{8}x^8+\tfrac{5}{16}x^{12}\ldots)dx=\theta+\frac{\theta^5}{10}+\frac{\theta^9}{24}+\tfrac{5}{208}\theta^{13}.$$

On substituting $\theta=\frac{1}{2}$ we get .5032 for A_θ as before.

II. **Solution by the PROPOSER.**

Let $x=\tan\omega$. Then $dx=\frac{d\omega}{\cos^2\omega}$. Substituting these values in the integral expression, we have

$$A_\theta=\int_0^\omega\frac{d\omega}{\cos^2\omega\sqrt{[(1-\tan^2\omega)(1+\tan^2\omega)]}}$$

$$=\int_0^\omega\frac{d\omega}{\sqrt{[\cos^2\omega-\sin^2\omega]}}=\int_0^\omega\frac{d\omega}{\sqrt{[1-2\sin^2\omega]}},$$

where $\tan\omega=\theta$ is used for the upper limit.

Now let $\sqrt{2}\sin\omega=\sin\phi$; then $1-2\sin^2\omega=\cos^2\phi$ and

$$d\omega=\frac{\cos\phi\, d\phi}{\sqrt{2}\sqrt{[1-\frac{1}{2}\sin^2\phi)]}}.$$

Whence $A_\theta=\frac{1}{2}\sqrt{2}\int_0^\phi \frac{d\omega}{\sqrt{[1-\frac{1}{2}\sin^2\phi]}}=\frac{1}{2}\sqrt{[2]}F(\frac{1}{2}\sqrt{2},\ \phi)$.

The amplitude ϕ is determined from $\theta=\tan\omega$, $\cos^2\omega=\frac{1}{1+\theta^2}$, $\sin^2\omega=\frac{\theta^2}{1+\theta^2}$, $\sin^2\phi=\frac{2\theta^2}{1+\theta^2}$, $\cos^2\phi=\frac{1-\theta^2}{1+\theta^2}$. Hence, $\tan^2\phi=\frac{2\theta^2}{1-\theta^2}$.

Referring to problem 303,

$$A=\int_0^1 \frac{dx}{\sqrt{[1-x^4]}}=\frac{\sqrt{\pi}}{4}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})},$$

the well known formula $\Gamma(\theta)\Gamma(1-\theta)=\pi/\sin\pi\theta$, gives in our case $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})=\pi\sqrt{2}$. Whence, $\Gamma(\frac{3}{4})=\frac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})}$, and therefore, $A=\frac{[\Gamma(\frac{1}{4})]^2}{4\sqrt{[2\pi]}}$.

This combined with the result in above, $A=\frac{1}{2}\sqrt{[2]}F^I(\frac{1}{2}\sqrt{2})$ yields $\Gamma(\frac{1}{4})=2\sqrt[4]{[\pi]}\{F^I(\frac{1}{2}\sqrt{2})\}^{\frac{1}{2}}=3.62561$, $^*0.5593811$.

And similarly, $\Gamma(\frac{3}{4})=\frac{1}{2}\sqrt{[2]}\sqrt[4]{[\pi^3]}\{F^I(\frac{1}{2}\sqrt{2})\}^{-\frac{1}{2}}=1.225416$, $^*0.0882838$.

These expressions for $\Gamma(\frac{1}{4})$ and $\Gamma(\frac{3}{4})$ in Legendre's F-functions are to my mind by far the most important consequences of evaluating integral A in gamma-functions. Without this evaluation $\Gamma(\frac{1}{4})$ and $\Gamma(\frac{3}{4})$ can be determined only by computing their natural logarithms by inconvenient series.

Also solved by V. M. Spunar, C. N. Schmall, and J. Scheffer.

MECHANICS.

253. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

R_1 and R_2 are ranges on a horizontal plane of particles projected with given velocity from A on the plane to pass through B. Show that $a(R_1+R_2)-R_1R_2=\frac{a^4}{c^2}$, where $c=AB$ and a is the horizontal projection of AB.

III. Solution by the PROPOSER.

If α, α_1 be angles of projection and β the angle AB makes with the horizontal, and v the velocity of projection, then $\alpha_1=\frac{1}{2}\pi-(\alpha-\beta)$.

And $\cos\beta=a/c$; $R_1.g=a^2\sin 2\alpha$; $R_2.g=a^2\sin 2(\alpha-\beta)$.

$$AB=c=\frac{2v^2}{g}\frac{\cos\alpha\sin(\alpha-\beta)}{\cos\beta}=\frac{2v^2c^2}{g\,a^2}\sin(\alpha-\beta)\cos\alpha.$$

$$\therefore a(R_1+R_2)=\frac{av^2}{g}[\sin 2\alpha+\sin 2(\alpha-\beta)]=\frac{2av^2}{g}\sin 2(\alpha-\beta)\cos\beta$$

$$=\frac{2av^2}{g}[\sin\alpha\cos(\alpha-\beta)+\cos\alpha\sin(\alpha-\beta)]\cos\beta$$

$$=\frac{a^4}{c^2}\left(\frac{\sin\alpha\cos(\alpha-\beta)+\cos\alpha\sin(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}\right).$$

$$R_1R_2=\frac{v^4}{g^2}\sin 2\alpha\sin 2(\alpha-\beta)=\frac{4v^4}{g^2}\sin\alpha\cos\alpha\sin(\alpha-\beta)\cos(\alpha-\beta)$$

$$=\frac{a^4}{c^2}\frac{\sin\alpha\cos(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}.$$

$$\therefore a(R_1+R_2)-R_1R_2=\frac{a^4}{c^2}\frac{\cos\alpha\sin(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}=\frac{a^4}{c^2}.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

Edited by Dr. G. E. Wahlin, University of Illinois.

183. Proposed by MERTON T. GOODRICH, Dixfield, Maine.

What relations must exist between the quantities A, B, and C in the harmonic ratio $\frac{AB}{(A+B+C)(-C)}=-1$ so that they will be positive integers.

II. Solution by the PROPOSER.

Solving the given equation for A, we have $A=\frac{(B+C)C}{B-C}$. Since A, B, and C are positive, we see that $B-C$ must be positive. Since A is an integer, either $\frac{C}{B-C}=\frac{M}{N}=K$, or $\frac{B+C}{B-C}=\frac{M'}{N'}=K'$, where M and N, M' and N' are integers, and M prime to N, and M' prime to N'. From the first of these equations, $C=K(B-C)$. Since C and $B-C$ are positive, K must be positive. Solving this last equation for B, we have $B=\frac{(K+1)C}{K}=\frac{(M+N)C}{M}$. Since M and N are relatively prime, $M+N$ is prime to M. Hence, B being an integer, C is divisible by M. That is, $C=MD'=MND'/N=KD$, where $D=ND'$, and D' and hence D are positive integers.

Substituting KD for C in the expression for B, $B=(K+1)D=C+D$. Substituting KD for C and $(K+1)D$ for B in the expression for A, $A=(2K+1)KD=(2K+1)C$. Putting $\frac{M}{N}$ for K, $A=\frac{(2M+N)MD}{N^2}$. This tells us that if N is odd $D=N^2.\lambda$; but if N is even, $2M+N$ is even and then $D=\frac{N^2.\lambda}{2}$. Hence we have this set of relations: $A=(2K+1)C$, $B=C+D$,

$C=KD$, where K is a positive fraction in its lowest terms or a positive integer, and D is a positive integer as above determined.

The second condition is not independent of the first, because, if K' is substituted for $2K+1$ and $2D'$ for D, in the above set of relations, then we have

$$\frac{B+C}{B-C}=\frac{D'K'+D+'D'K'-D'}{D'K'+D'-D'K'+D'}=\frac{2D'K'}{2D'}=K',$$

which is the second condition. Hence the set of values found above includes all the possible relations which make A, B, and C positive integers. The values of A and B may be interchanged, and A, B, and C may each be multiplied by a common factor without changing the value of the original ratio.

Also solved by A. H. Holmes.

184. **Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.**

Prove that $\frac{\pi}{12}=\tan^{-1}\frac{1}{2^2}+\tan^{-1}\frac{1}{8^2}+\tan^{-1}\frac{1}{30^2}+\tan^{-1}\frac{1}{112^2}+\ldots$, where 2, 8, 30, 112..., is a recurring series with the recursion formula $u_n=4u_{n-1}-u_{n-2}$.

Solution by the PROPOSER.

If we reduce $\sqrt{3}$ to a continued fraction, we get for the convergents,

$$\frac{1}{1},\ \frac{2}{1},\ \frac{5}{3},\ \frac{7}{4},\ \frac{19}{11},\ \frac{26}{15},\ \frac{71}{41},\ \frac{97}{56},\ \frac{265}{153},\ \frac{362}{209},\ \ldots$$

The alternate convergents,

$$\frac{1}{1},\ \frac{5}{3},\ \frac{19}{11},\ \frac{71}{41},\ \frac{265}{153},\ \ldots$$

are formed by taking the ratios of the corresponding terms of the two recurring series

$$1,\ 5,\ 19,\ 71,\ 265,\ \ldots$$
$$1,\ 3,\ 11,\ 41,\ 153,\ \ldots$$

both having the same scale of relation $u_n=4u_{n-1}-u_{n-2}$.

We find by the usual methods that the nth terms of the two series are

$$\frac{\alpha^n-\beta^n}{\alpha-\beta}+\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}=u_n+u_{n-1},\ \text{and}\ \frac{\alpha^n-\beta^n}{\alpha-\beta}-\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}=u_n-u_{n-1},$$

where α and β are the roots of the equation $x^2-4x+1=0$.

Taking the differences of the $\tan^{-1}$ function of the reciprocals of the convergents, we have

$$\tan^{-1}\frac{1}{1}-\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{1}{4}$$

$$\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{11}{19}=\tan^{-1}\frac{1}{64}$$

$$\tan^{-1}\frac{11}{19}-\tan^{-1}\frac{41}{71}=\tan^{-1}\frac{1}{900}$$

$$\tan^{-1}\frac{41}{71}-\tan^{-1}\frac{153}{265}=\tan^{-1}\frac{1}{12544}$$

$$\cdots\cdots\cdots\cdots$$

$$\tan^{-1}\frac{u_n-u_{n-1}}{n_n+u_{n-1}}-\tan^{-1}\frac{u_{n+1}-u_n}{u_{n+1}+u_n}=\tan^{-1}\frac{1}{4u_n^2}$$

$$\cdots\cdots\cdots\cdots$$

Adding, we have

$$\tan^{-1}\frac{1}{1}-\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{4}-\frac{\pi}{6}=\frac{\pi}{12}=\tan^{-1}\frac{1}{2^2}+\tan^{-1}\frac{1}{8^2}+\tan^{-1}\frac{1}{30^2}+\ldots$$

$$+\tan^{-1}\frac{1}{(2u_n)^2}+\ldots$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

360. **Proposed by CHARLES C. GROVE, Columbia University, New York.**

A bridge club of 28 members has 27 meetings. There are 7 tables with 4 members at each table. Can the players be so arranged that at the end of the season (27 meetings) each member will have played *with* every other member one game and *against* every other member two games, one game meaning one meeting; and how?

361. **Proposed by C. E. GITHENS, Ph. D., Wheeling, W. Va.**

Find three integral values for $[-10+9\sqrt{(-3)}]^{1/3}+[-10-9\sqrt{(-3)}]^{1/3}$. A solution not involving a cubic is desired.

362. **Proposed by JAMES F. LAWRENCE, Stillwater, Okla.**

Show that the number of solutions in positive integers, zero included, of the equation $x+2y+3z=6n$, is $3n^2+3n+1$.

GEOMETRY.

393. **Proposed by S. LEFSEHETZ, Clarke University.**

Draw a triangle having a given angle, and with its vertices on three given concentric circles.

394. **Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.**

The joins of the excentres to the corresponding vertices of the pedal triangle are concurrent.

395. **Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.**

From a point P without a rectangular field ABC the distances PA, PB, PC measured to the corners are respectively 70, 40, 60 chains. What is the area of the field?

CALCULUS.

316. **Proposed by C. N. SCHMALL, New York City.**

$$\int_0^\infty \frac{\cos ax}{1+x^2}dx=\tfrac{1}{2}\pi e^{-a}=\int_0^\infty \frac{x\sin ax}{1+x^2}dx.$$

(From Bromwich, *Theory of Infinite Series*, p. 442, ex. 5, and also from Carslaw, *Fourier's Series*, p. 113, ex. 12.) Prove this by any method.

317. **Proposed by C. N. SCHMALL, New York City.**

A generating line of a right circular cylinder passes through the center of a sphere. The diameter of the cylinder is less than the radius of the sphere. Show that the surface of the cylinder included within the sphere is given by an elliptic integral.

MECHANICS.

263. **Proposed by C. N. SCHMALL, New York City.**

A railroad car is rounding a curve of radius r with a velocity v, $2d$ being the distance between the rails. If h be the height of its center of gravity above the rails, and g have its usual meaning, show that the weight of the car is divided between the outer and inner rails in the ratio $\dfrac{dgr+v^2h}{dgr-v^2h}$.

264. **Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.**

Three particles, weights ω_1, ω_1, ω_2, and three light strings of equal length connecting them, lie in a vertical smooth circular tube. Discuss the possible positions of stable and unstable equilibrium. If a slight displacement takes place in the unstable position, find the maximum ensuing velocity.

NOTES AND NEWS.

The new Spanish Mathematical Society, of Madrid, Spain, has issued the third number of its journal called *Revista de la Sociedad Matematica Espanola*. According to a statement on page 96 of this journal the membership of this society has grown very rapidly, having surpassed 370 on the date of the first reunion which was held in Madrid on the 28 day of last June. M.

B. G. Teubner, of Leipzig, Germany, has published the first volume of *Euler's Complete Works*. It has been estimated that these works would comprise about forty large volumes and that the cost of publication would be about eighty thousand dollars. It may be remembered that the people of Switzerland and various learned societies, including the American Mathematical Society, contributed liberally towards this publication. Euler has been the most prolific mathematical writer up to the present time, and his works have had a very important influence on the development of mathematics during almost two centuries. This first volume is in German and deals with elementary arithmetic and elementary algebra. M.

A recent number of the *Rendiconti del Circolo Matematico di Palermo* contains the following interesting statistics in reference to the principal mathematical societies existing in 1911. The largest of these is the Circolo Matematico di Palermo, founded in 1884, and having a membership of 810 in August of the present year. Only a little more than one-third of these members, 286, resided in Italy. The second society in order of membership is the Deutsche Mathematiker-Vereinigung, founded in 1890, and having a membership of 751 at the beginning of the present year. Considerably more than one-half of these, 449, resided in the empire of Germany. The American Mathematical Society, founded in 1888, and having a membership of 641 at the beginning of the present year, is the third in order of size. As only 53 of these members resided outside of the United States, this society has a considerably larger resident membership than any other. The oldest society is the Mathematische Gesellschaft in Hamburg, founded in 1690, and having a membership of 111 at the beginning of the present year. M.

BOOKS.

Monographs on Topics of Modern Mathematics Relevant to the Elementary Field. Edited by J. W. A. Young. 8vo. viii+416 pages. Price, $3.00. New York: Longmans, Green & Co.

Titles and Authors.—I. The Foundation of Geometry. By Oswald Veblen, Ph. D., Professor of Mathematics in Princeton University. II. Modern Pure Geometry. By Thomas F. Holgate, Ph. D., LL. D., Professor of Mathematics in Northwestern Univer-

sity. III. Non-Euclidean Geometry. By Frederick S. Woods, Ph. D., Professor of Mathematics in the Massachusetts Institute of Technology. IV. The Fundamental Propositions of Algebra. By Edward V. Huntington, Ph. D., Assistant Professor of Mathematics in Harvard University. V. The Algebraic Equation. By G. A. Miller, Ph. D., Professor of Mathematics in the University of Illinois. VI. The Function Concept and the Fundamental Notions of the Calculus. By Gilbert Ames Bliss, Ph. D., Associate Professor of Mathematics in the University of Chicago. VII. The Theory of Numbers. By J. W. A. Young, Ph. D., Associate Professor of the Pedagogy of Mathematics in the University of Chicago. VIII. Constructions with Ruler and Compasses; Regular Polygnons. By L. E. Dickson, Ph. D., Professor of Mathematics in the University of Chicago. IX. The History and Transcendence of π. By David Eugene Smith, Ph. D., LL. D., Professor of Mathematics in Teachers College, Columbia University.

Editor's Preface. "The purpose of this collection of monographs may be indicated by the following citation from the letter that was sent to those who were requested to act as authors.

"Among the varions publications on mathematics that are being made, it would seem that there is room for a serious effort to bring within reach of secondary teachers (in service or in training), college students, and others at a like stage of mathematical advancement, a scientific treatment of some of the regions of advanced mathematics that have points of contact with the elementary field. Undoubtedly one of the most crying needs of our secondary instruction in mathematics to-day, is that the scientific attainments of the teachers be enlarged and their mathematical horizon widened; and I believe that there is a large body of earnest teachers and students that are eager to extend their mathematical knowledge if the path can be made plain and feasible for them.

"A volume of monographs dealing with selected topics of higher mathematics might well be a useful contribution to the meeting of this need. Such monographs would aim to bring the reader into touch with some characteristic results and viewpoints of the topics considered, and to point out their bearings on elementary mathematics. They would therefore contain:

(1) A considerable body of results proved in full, so that the reader can materially extend his mathematical acquisitions by the reading of the monograph alone.

(2) Statement without proof of some leading methods and results, so as to give a bird's-eye view of the subject.

(3) A small number of references indicating what the reader may profitably take up after he has mastered the contents of the monograph."

Both the plan itself, and the invitation to act as author, were most cordially received; work on the monographs was promptly begun, has been carried through substantially as planned, and the results are now presented in this book."

Vocational Algebra. By George Wentworth and David Eugene Smith. 12mo. Cloth. 88 pages, Illustrated. Price, 50 cents. New York and Chicago: Ginn & Co.

"The time has arrived when algebraic language has such a well-defined place in trade journals, artisans' manuals, and handbooks of business that the workman in the shop and the business man in the office have each a practical need to interpret it. The growth of industrial and commercial classes, one of the most significant features of our present work in education, helps to create such a need, and the shop itself is not far behind the school in making it known. It is to meet the demand for the essentials of algebra thus required in preparation for the shop and commerce that 'Vocational Algebra' has been written. It presents exactly the topics that are needed in vocational classes, no more and no less. Any one who has mastered it will be able to understand all the algebra of ordinary trade or business. The book contains a wide range of general vocation problems, but is free from mere puzzles and from those technicalities with which neither teacher nor stu-

dent should be expected to be familiar. In addition to this applied work it offers a sufficient amount of drill in abstract algebraic forms to insure a mastery of all the principles involved.

"Throughout the formula and equation are continually reviewed and constitute the leading feature of every chapter."

The Elements of Plane and Spherical Trigonometry. By John Gale Hun and Charles Ranald MacInnes. 8vo. Cloth, 100 pages of Text, 92 pages of Tables, and 13 pages of Explanation of Tables. Price, $1.35. New York: The Macmillan Co.

The material in this book has been used in pamphlet form from three to four years in Princeton University. The authors have attempted to present the essentials of subject in as brief and clear a manner as possible. Some space is devoted to the drawing of graphs of simple equations in polar coördinates because it is thought that such problems aid the student in getting a clearer idea of the way in which the functions vary with the change of angle. Many of the proofs are very brief though easily understood by the average student. F.

The Hindu-Arabic Numerals. By David Eugene Smith, Professor of Mathematics, Teachers College, Columbia University, and Louis C. Karpinski, Instructor in Mathematics, University of Michigan, Ann Arbor. 8vo. Cloth, 160 pages. Price, $1.25. Boston: Ginn & Co.

"Although it has long been known that the numerals ordinarily employed in business and commonly attributed to the Arabs, are not of Arabic origin, and although numerous monographs have been written concerning their derivation, no single work has yet appeared in which the complete story of their rise and development has been told. In the preparation of this treatise the authors have examined every important book and monograph that has appeared upon the subject, consulting the principal libraries of Europe as well as America, examining many manuscripts, and sifting the evidence with greatest care. The result is a scholarly discussion of the entire question of the origin of the numerals, the introduction of the zero, the influence of the Arabs, and the spread of the system about the shores of the Mediterranean and into the various countries of Europe.

The work is illustrated with numerous facsimilies from early inscriptions and manuscripts, most of which have not heretofore been published in connection with this subject, and all of which contribute to a very marked degree to an understanding of the problem.

Such a contribution to history, to mathematics, and to education bearing the names of two authors of such prominence in the history of mathematics, should find a place in every library of importance, and upon the shelves of all who are interested in education in its broadest aspect."

The Progress of Physics, During the 33 years (1875-1908). Four Lectures Delivered to the University of Calcutta During March, 1908. By Arthur Schuster, F. R. S., Ph. D., (Heidelberg), Sc. D. (Cantab), D. Sc., (Manchester and Calcutta), Dès. Sc. (Geneva).

8vo. Cloth, x+104 pages. Price $1.25. Cambridge, England: The University Press: G. P. Putnam's Sons, American Agents.

When one has once begun the reading of these four lectures, one will not want to lay the book aside until its perusal is completed. The author's personal contact and acquaintance with the greatest scientist of the past thirty years, makes his descriptions of their experiments and discoveries of more than ordinary value. The frontispiece is an excellent portrait of James Clerk-Maxwell. F.

A Treatise on Electric Theory and the Problem of the Universe. Considered from the Physical Point of View, with Mathematical Appendices. By G. W. de Tunzelmann, B Sc., Member of the Institution of Electrical Engineers, Formerly Professor of Physics and Astronomy, H. M. S. "Britannia," Dartmouth. 8vo. Cloth. xxxi+654 pages. Price, $4.50. Philadelphia: J. B. Lippincott & Co.

This volume is profoundly interesting and instructive. It deals with the most absorbing questions in the whole realm of matter. Some idea of the range of subjects discussed may be learned from the subjects of the twent-four chapters into which the book is divided. Chapter I, Fundamental Electrical Phenomena; chapter II, Units and Measurements; chapter III, Meaning and Possibility of a Mechanical Theory of Electricity; chapter IV, The Ether; chapter V, The Ether as a Framework for Absolute Motion; chapter VI, Relations Between Ether and Moving Matter; chapter VII, Conditions in Gases and Dielectrics; chapter VIII, The Faraday-Maxwell Theory; chapter IX, The Electron Theory; chapter X, Magnetism and the Dissipation of Energy; chapter XI, Contact Electrification and Electrolysis; chapter XII, Optical Phenomena; chapter XIII, The Mechanism of Radiation; chapter XIV, Metallic Conduction and Radiation; chapter XV, General Phenomena of Radioactivity; chapter XVI, The Three Principal Types of Radioactivity; chapter XVII, Transformation of Radioactive Substances; chapter XVIII, The Ages of the Earth and Sun, and the Probable Origin of Radio-active Substances; chapter XIX, The Solar Corona, the Aurora and Comets' Tails; chapter XX, Radio-activity in Stars and Nebulae; chapter XXI, Arrangments and number of Electrons in an Atom; chapter XXII, Change in the Aspect of Fundamental Mechanical Principles; chapter XXIII, Gravitation and Cohesion; chapter XXIV, The Place of Mind in the Universe. While one may not accept all the theories advanced in this volume, one cannot help being interested in them, for further investigation and study may raise them to the rank of definite laws and principles. F.

Optical Geometry of Motion. A New View of the Theory of Relativity. Alfred A. Robb, M. A., Ph. D. Pamphlet, 32 pages. Cambridge: W. Heffer & Sons.

An essay which puts the theory of Relativity in mathematical dress. F.

Elements of Trigonometry. By Daniel A. Murray, Ph. D., Professor of Applied Mathematics in McGill University. 8vo. Cloth. ix+136 pages. Price, 75 cents. New York: Longmans, Green and Co.

This is a shorter course than the author's former text-book entitled *Plane Trigonometry.* The omission of many notes and of several topics in the former and a more condensed treatment of others enables him to present the essentials in this abbreviated form. F.

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THE TENTH PERFECT NUMBER.

By R. E. POWERS, Denver, Colorado.

A number which is equal to the sum of all its divisors is called a "perfect number." Thus, the divisors of 6 are 1, 2, and 3, the sum of which is equal to 6; the divisors of 28 are 1, 2, 4, 7, and 14, whose sum is 28. Euclid (IX, 36) proved that if 2^p-1 is prime, then $2^{p-1}(2^p-1)$ is a perfect number, and no other perfect numbers are known. In 1644 Mersenne, in the preface to his *Cogitata Physico-Mathematica*, stated, in effect, that the only values of p not greater than 257 which make 2^p-1 prime are 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, and 257. Regarding these "Mersenne's Numbers" (2^p-1), W. W. Rouse Ball, in his *Mathematical Recreations and Essays* (4th Edition, pages 262, 263, 269), says:

"I assume that the number 67 is a misprint for 61. With this correction, we have no reason to doubt the truth of the statement, but it has not been definitely established. . . . Seelhoff showed that 2^p-1 is prime when $p=61$, . . . and Cole gave the factors when $p=67$. . . . One of the unsolved riddles of higher arithmetic . . . is the discovery of the method by which Mersenne or his contemporaries determined values of p which make a number of the form 2^p-1 a prime. . . . The riddle is still, after nearly 250 years, unsolved."

No exception to Mersenne's assertion (corrected by the substitution of 61 for 67) is known at the present time. Below we show, however, that $2^{89}-1$ is a prime number, contrary to his statement.

Following is a list of Mersenne's Numbers thus far proved to be prime, with the corresponding perfect numbers:

p	2^p-1	*Perfect Numbers*
2	3	6
3	7	28
5	31	496
7	127	8,128
13	8,191	33,550,336
17	131,071	8,589,869,056
19	524,287	137,438,691,328
31	2,147,483,647	(19 digits)
61	2,305,843,009,213,693,951	(37 digits)

To these must now be added the prime number $2^{89}-1$, so that the tenth perfect number is $2^{88}(2^{89}-1)$, or

$$191561942608236107294793378084303638130997321548169216$$

(it is known that 2^p-1 is composite for all other values of p not greater than 100).

In his *Théorie des Nombres,* page 376, Lucas says: "Nous pensons avoir démontré par de très longs calculs qu'il n'existe pas de nombres parfaits pour $p=67$ et $p=89$." While this result has since been verified for $p=67$, the opinion has been expressed that also the case $p=89$ needed an independent examination. The result here shown that $2^{89}-1$ is a prime is therefore in conflict with Lucas' computation. The same writer, in an article entitled "Théorie des Fonctions Numériques Simplement Périodiques," Section XXIX, in the *American Journal of Mathematics,* Volume 1 (1878), proved the following remarkable theorem (the theorem appears on page 316 of the volume):

"If $P=2^{4q+1}-1$, and we form the series of residues (modulo P)

$$4,\ 14,\ 194,\ 37634,\ \ldots,$$

each of which is equal to the square of the preceding, diminished by two units: the number P is composite if none of the $4q+1$ first residues is equal to O; P is prime if the first residue O lies between the $2q$th and the $(4q+1)$th term."

Applying the above theorem to the number $2^{89}-1$, and denoting the terms of the series by L_1, L_2, L_3, ..., we found the following residues (modulo $2^{89}-1$):

m	L_m
1	4
2	14
3	194
. . .	. . . .
10	—115,113,975,804,653,882,052,836,464
20	36,000,517,785,442,762,303,479,300
30	—204,144,540,641,167,292,618,604,303
40	−126,791,709,316,676,382,795,042,761
50	−90,990,560,635,837,660,454,542,648
60	−206,308,592,424,355,282,693,419,690
70	99,498,791,857,820,493,810,407,653
80	269,783,273,665,984,523,074,966,550
. . .	. . . .
86	—309,403,333,482,440,150,628,882,422
87	−35,184,372,088,832
88	0

Since the first (and only) residue 0 occurs at the 88th term of the above series, it follows, from the foregoing theorem, that $2^{89}-1$, or

618,970,019,642,690,137,449,562,111

is a PRIME NUMBER.

As M. Lucas points out, his method used above is free from any uncertainty as to the accuracy of the conclusion that the number under consideration is prime, in case our attempt to arrive at the residue 0 meets with success, since an error in calculating any term of the series would have the effect of preventing the appearance of the residue 0. We would add that, denoting the number $2^{89}-1$ by N, we have verified that

$3^{N-1}-1$ is divisible by N,

which is in accordance with Fermat's well-known theorem.

DENVER, COLORADO, *June, 1911.*

A GENERALIZATION OF CAUCHY'S FUNCTIONAL EQUATIONS.*

By R. D. CARMICHAEL, Indiana University.

§ 1. INTRODUCTION.

Cauchy† has determined the general continuous solution of each of the simple functional equations

$$
\begin{aligned}
f(x+y)&=f(x)+f(y),\\
f(x+y)&=f(x)\,.\,f(y),\\
f(xy)&=f(x)\,.\,f(y),\\
f(xy)&=f(x)+f(y),
\end{aligned}
$$

showing that they are, respectively, ax, a^x, x^a, $a\log x$, where in each case a is an arbitrary constant.

Several generalizations of these results have been given. For instance, in the case of the first equation it has been shown‡ that the conclusion remains the same if the hypothesis is weakened by requiring that $f(x)$ shall be continuous only in a finite interval, however small. Or, one may require§ merely that the function shall have an upper limit in an arbitrarily small neighborhood of $x=0$. A generalization of a different kind has been given by H. W. Pexider.||

In the present paper a generalization of a different type is introduced. Instead of two variables, x and y, of unlimited variation, we retain one variable x of unlimited variation and replace the variable y by a function u_z of the variable z which has the property that, for some set of values of z, u_z may assume at least once each value of an assigned enumerably infinite set of values. Otherwise the function u_z is entirely arbitrary; it need not even be defined at other points. Furthermore, $f(x)$ is required to be continuous only in a given finite interval. The general solution remains the same as before.

This generalization is interesting in that it brings out explicitly the fact that in previous discussions of the problem, as for instance by Cauchy and Darboux, the hypothesis was much stronger than was necessary to enable one to draw the conclusion.

The method employed in the present note (see § 2) is capable of wide use and extension in the study of functional equations of various types. It is due to Monge and Laplace. It consists, abstractly, in reducing the original problem to two steps. The first step consists in finding the general so-

* Read before the American Mathematical Society, October 28, 1911.

† Cauchy, *Œuvres* (2) 3 (1897), p. 98, p. 220.

‡ Darboux, *Mathematische Annalen* 17 (1880), p. 55; Segre, Torino Atti 25 (1890), p. 192, p. 287.

§ La Vallée-Poussin, *Course d'analyse*, 1903, p. 30.

|| *Monatshefte fur Mathematik und Physik*, 14 (1902), p. 293.

lution of a difference equation which is a special case of the given equation, or is readily deducible from it, this equation having the property that some particular solution of it is the general solution of the original functional equation. The second step consists in determining the arbitrary periodic function or functions of this solution of the difference equation so as to obtain that special solution which satisfies the given functional equation. A like use may also be made of q-difference equations in the solution of functional equations.

In the present paper all quantities are supposed to be real. The extension to the case of complex quantities is not difficult.

§ 2. THE EQUATION $f(x+u_z)=f(x)+f(u_z)$.

We seek to determine completely the function $f(x)$ subject to the conditions that it is to be continuous in the interval 0 to α, where α is any positive or negative constant whatever, and that it is to satisfy the functional equation

$$f(x+u_z)=f(x)+f(u_z), \tag{1}$$

where u_z is a function of the variable z subject to the sole restriction that it is capable of assuming at least once each value in the set

$$2^{-s}n\,\alpha;\quad n,\ s=0,\ 1,\ 2,\ \ldots,\quad n \overline{\overline{<}}\, 2^s,$$

$u_z=z$ is the simplest such function; and hence equation (1) is a generalization of the Cauchy equation

$$f(x+z)=f(x)+f(z).$$

But the possible functions u_z are evidently of the most varied character. The most general definition of u_z is the following:

Let z_{ns}, n, $s=0$, 1, 2, ..., $n \overline{\overline{<}}\, 2^s$, be any set of numbers no two of which are equal and let

$$u_z=\frac{n\,\alpha}{2^s}\ \text{for}\ z=z_{ns};$$

otherwise let u_z be arbitrary. It need not even be defined for other values of z. Under these conditions we shall show that the solution of (1) is unique except for an arbitrary constant multiplier; in fact, that it is

$$f(x)=ax,$$

where a is an arbitrary constant.

Give to z a value for which u_z is equal to α; then from (1) we have the difference equation

$$f(x+\alpha)=f(x)+f(\alpha). \tag{2}$$

Therefore the general solution of (1) must be some particular solution of (2). But the general solution of (2) is readily found. It is evidently

$$f(x)=ax+p(x) \tag{3}$$

where $a=f(\alpha)/\alpha$ and $p(x)$ is an arbitrary periodic function of period α. Since $f(x)$ is continuous in the interval $(0, \alpha)$, $p(x)$ is continuous in $(0, \alpha)$. Substiiuting this value of $f(x)$ in (1) and reducing, we have

$$p(x+u_z)=p(x)+p(u_z), \tag{4}$$

an equation of the same type as (1) but subject to the further condition that its solution is to be periodic of period α.

Choose z so that $u_z=0$ and take $x=0$. Then from (4) we have

$$p(0)=2p(0); \text{ or}$$
$$p(0)=0,$$

since $p(x)$ is finite at $x=0$. Hence

$$0=p(0)=p(\alpha)=p(2\alpha)=\ldots=p(n\alpha).$$

Give to z a value such that u_z takes on the value $2^{-s}n\alpha$ and let $x=2^{-s}n\alpha$. Then from (4) we have

$$p\left(\frac{n\alpha}{2^{s-1}}\right)=2p\left(\frac{n\alpha}{2^s}\right).$$

If we employ this as a recursion formula and remember that $p(n\alpha)=0$, we have

$$0=p\left(\frac{n\alpha}{2}\right)=p\left(\frac{n\alpha}{2^s}\right)=\ldots=\left(\frac{n\alpha}{2^s}\right).$$

That is, the function $p(x)$ is equal to zero at every point of the interval

$(0, \alpha)$ which is representable in the form $2^{-s}n\,\alpha$. But the set of points $2^{-s}n\,\alpha$, n, $s=0, 1, 2, \ldots, n \overline{\overline{<}} 2^s$ is everywhere dense in the interval $(0, \alpha)$; that is, in every neighborhood (however small) of any point $\bar{\alpha}$ of the interval $(0, \alpha)$ there is a point of the set $2^{-s}n\,\alpha$ different from $\bar{\alpha}$.

Now, if a function is continuous at every point of an interval $(0, \alpha)$ and is equal to zero at every point of a set of points everywhere dense on $(0, \alpha)$, it is clear that the function is equal to zero at every point of the interval $(0, \alpha)$. Hence $p(x)=0$ at every point of the interval $(0, \alpha)$. But $p(x)$ is a periodic function of period α; and therefore $p(x)$, being zero in an interval of length α, is identically zero. Substituting this value of $p(x)$ in (3) we have as the most general solution of (1),

$$f(x)=ax,$$

where a is an arbitrary constant. This result may be stated in the form of the following theorem:

THEOREM I. *Let α be any positive or negative qaantity and let u_z be any function of z capable of assuming at least once each value in the set*

$$2^{-s}n\,\alpha,\ n,\ s=0, 1, 2, \ldots, n \overline{\overline{<}} 2^s.$$

Then the most general function $f(x)$ which is continuous on the interval $(0, \alpha)$ and satisfies the functional equation

$$f(x+u_z)=f(x)+f(u_z)$$

is

$$f(x)=ax,$$

where a is an arbitrary constant.

§ 3. THE REMAINING CAUCHY FUNCTIONAL EQUATIONS.

The results of the last section we shall now apply to the problem of finding the most general solution of each of the following equations:

$$f(x+u_z)=f(x)f(u_z), \tag{5}$$

$$f(xu_z)=f(x)f(u_z), \tag{6}$$

$$f(xu_z)=f(x)+f(u_z), \tag{7}$$

where the functions are further restricted in each case to satisfy appropriate conditions of continuity on a limited interval. As a special value of u_z we may take $u_z=z$, so that (5), (6), and (7) are generalizations of Cauchy's equations

$$f(x+z)=f(x)f(z),$$
$$f(xz)=f(x)f(z),$$
$$f(xz)=f(x)+f(z).$$

If we take the logarithm of each member of (5) and write $g(v)$ for $\log f(v)$, we have

$$g(x+u_z)=g(x)+g(u_z).$$

Now if $g(x)$ is to be continuous in the interval $(0, \alpha)$ the conditions of theorem I are satisfied for the last equation, so that the general solution is

$$g(x)=ax,$$

where a is an arbitrary constant. Hence

$$\log f(x)=ax.$$

Therefore, $$f(x)=e^{ax}=\beta^x,$$

where β is an arbitrary constant. If $g(x)$ is continuous in the interval $(0, \alpha)$, $f(x)$ is continuous in the same interval, and conversely, since $f(x)$ cannot be zero on the interval without being identically zero (as the original equation shows). Hence from theorem I we have the following theorem:

THEOREM II. *If α and u_z are defined as in theorem I, then the most general function $f(x)$ which is continuous on the interval* $(0, \alpha)$ *and satisfies the functional equation*

$$f(x+u_z)=f(x)f(u_z)$$

is $$f(x)=\beta^x,$$

where β is an arbitrary constant.

In equation (6) let us write

$$x=e^y,\quad u_z=e^{v_z},\quad f(e^t)=g(t);$$

then we have $$f(e^{y+v_z})=f(e^y)f(e^{v_z});$$

or $$g(y+v_z)=g(y)g(v_z).$$

If we assume that v_z satisfies the conditions imposed on u_z in the two previous cases, and that $g(y)$ is continuous in the interval $(0, \alpha)$, we have the hypotheses of theorem II reproduced. Consequently that theorem can be applied to the last equation. If this is done and if the result is stated

in terms of the original function f and the variables x and u_z we have the theorem:

THEOREM III. *Let* α *be any positive or negative quantity and let* u_z *be any function of* z *capable of assuming at least once each value in the set*

$$e^{2^{-s}n\alpha},\ n,\ s=0,\ 1,\ 2,\ \ldots,\ n \leqq 2^s.$$

Then the most general function $f(x)$ *which is continuous on the interval* $(1, e^{\alpha})$ *and satisfies the functional equation*

$$f(xu_z)=f(x)f(u_z)$$

is

$$f(x)=x^r,$$

where r *is an arbitrary constant.*

For, the most general value of $g(y)$ is

$$g(y)=\beta^y.$$

Hence,

$$f(e^y)=\beta_y, \text{ since } g(y)=f(e^y).$$

Now put $x=e^y$, whence $y=\log x$; then we have

$$f(x)=\beta^{\log x}=e^{r\log x},$$

where $r=\log\beta$. But $e^{r\log x}=e^{\log x^r}=x^r$. Hence the most general value of $f(x)$ is

$$f(x)=x^r,$$

where r is an arbitrary constant.

If now we take the logarithm of each member of (6) and write $g(v)$ for $\log f(v)$, we have

$$g(xu_z)=g(\alpha)+g(u_z),$$

an equation of the form (7). Hence from theorem III we have at once the following theorem:

THEOREM IV. *If* α *and* u_z *are defined as in theorem III, then the most general function of* $f(x)$ *which is continuous on the interval* $(1, e^{\alpha})$ *and satisfies the functional equation*

$$f(xu_z)=f(x)+f(u_z)$$

is

$$f(x)=r\log x,$$

where r *is an arbitrary constant.*

THE CYCLIC GROUP AS A BASIC ELEMENT IN THE THEORY OF NUMBERS.

By G. A. MILLER, University of Illinois.

The main objects of the present paper are to point out some of the uses of properties of the cyclic group in studying some of the fundamental theorems of number theory, and to develop these properties of the cyclic group by means of elementary arithmetic considerations, which relate mainly to additive number theory. It is hoped that the paper may serve the double purpose of furnishing an easy but rigorous arithmetic introduction to the cyclic group, and of presenting somewhat novel proofs of wide scope for a few very elementary theorems in number theory.

No knowledge of group theory is presupposed, and the only knowledge of number theory that is presupposed is the fact that a rational integer can be resolved into its prime factors in essentially only one way, together with some familiarity with the elementary notions of a congruence* and of a complete system of residues. Hence the substance and the mode of presentation of this paper would appear suitable for a second chapter of an elementary work on number theory. In fact, it presupposes only a very brief first chapter.

§ 1. ORDER OF A RATIONAL INTEGER (MOD m).

The term *order* of a rational integer a with respect to modulus m, m being a positive rational integer, will be used for the smallest positive rational integer b which is such that the product ab is divisible by m. In other words, the order of a (mod m) is equal to the least number of times that a must be taken as an addend in order to obtain a sum that is divisible by m, and hence the order of a is its period (mod m) as regards addition. It is evident that the order of a (mod m) is equivalent to the product of all the prime factors of m which are not also found in a, and hence *all the rational integers which are congruent to the integer a (mod m) have the same order (mod m).*

The m successive rational integers,

$$1,\ 2,\ 3,\ \ldots,\ m, \tag{A}$$

which represent the complete system of residues (mod m) composed of the smallest possible natural numbers, must therefore represent also numbers of all the possible orders (mod m). All these orders divide m and there is at least one number in (A) whose order is an arbitrary divisor of m.

* Two rational integers a, b are said to be congruent with respect to modulus m whenever $a-b$ is divisible by the rational integer m. This fact is denoted, according to Gauss, by $a \equiv b$ (mod m).

Suppose that $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_\lambda^{\alpha_\lambda}$; $p_1, p_2, \dots, p_\lambda$ being distinct rational prime numbers. A necessary and sufficient condition that a is of order $p_\beta^{\alpha_0}$ (mod m), $1 \overline{\overline{<}} \beta \overline{\overline{<}} \lambda$, is that a is a multiple of $m \div p_\beta^{\alpha_0}$ which is not divisible by $p_\beta^{\alpha_\beta - \alpha_0 + 1}$. There are, therefore, in (A) exactly $p_\beta^{\alpha_\beta}$ numbers whose orders are powers of p_β, viz., the numbers

$$\frac{m}{p_\beta^{\alpha_\beta}},\ 2\frac{m}{p_\beta^{\alpha_\beta}},\ 3\frac{m}{p_\beta^{\alpha_\beta}},\ \dots,\ p_\beta^{\alpha_\beta}\ \frac{m}{p_\beta^{\alpha_\beta}}.$$

We shall use n_β to represent any one of these $p_\beta^{\alpha_\beta}$ numbers, and hence the sum

$$n_1 + n_2 + n_3 + \dots + n_\lambda \qquad \text{(B)}$$

may represent any one of m numbers.

It is not difficult to prove that the totality of the possible numbers represented by (B) is a complete system of residues (mod m). To prove this it is only necessary to show that no two of these numbers are congruent (mod m). In fact, from the congruence

$$n_1 + \dots + n_\beta + \dots + n_\lambda \equiv n'_1 + \dots + n'_\beta + \dots + n'_\lambda \ (\text{mod } m).$$

it results, if $n_0 = n_{\lambda+1} = 0$, that

$$n_1 + \dots + (n_\beta - n'_\beta) + \dots + n_\lambda \equiv n'_1 + \dots + n'_{\beta-1} + n'_{\beta+1} + \dots + n'_\lambda \ (\text{mod } m).$$

As all these addends, except $n_\beta - n'_\beta$, are divisible by $p_\beta^{\alpha_\beta}$ this number must also divide $n_\beta - n'_\beta$ and hence $n_\beta = n'_\beta$, since all of these numbers may be supposed to be in (A). This proves the following theorem:

If the rational integer m is divisible by exactly λ distinct rational prime numbers, a complete system of residues (mod m) may be obtained by adding successively the numbers of each of the m possible different sets of λ numbers whose orders are powers of the λ different prime factors of m, the numbers having been selected from any complete system of residues (mod m).

To illustrate this theorem we let $m = 60 = 2^2.3.5$. The numbers of (A) whose orders are powers of 2, 3, and 5, respectively, constitute the following rows:

15, 30, 45, 60
20, 40, 60
12, 24, 36, 48, 60

It is not difficult to verify that the 60 different sums obtained by taking one and only one addend from each of these three rows constitutes a complete system of residues (mod 60).

Suppose that p^{α}_{β} is the highest power of p which divides the order of a (mod m), and that the order of b (mod m) is not divisible by p^{α}_{β}. It results directly from the definition of order that the highest power of p_{β} which divides a is ${}^{\alpha}_{\beta}{}^{\beta-\alpha}$ and that b is divisible by a higher power of p_{β}. Hence $a+b$ is divisible by $p^{\alpha}_{\beta}{}^{\beta-\alpha}$ but not by the $(\alpha_{\beta} - \alpha + 1)$th power of p_{β}, since the latter of these numbers divides b but does not divide a. This proves the theorem:

If p^{α} is the highest power of a rational prime number which divides the order (mod m) of one of two rational integers without dividing this order of the other, then the order (mod m) of the sum of these integers is divisible by p^{α} but not by $p^{\alpha+1}$.

From this theorem we can readily deduce the following useful corollary: *If the orders of two or more rational integers are relatively prime, the order of their sum is the product of their orders, all orders being taken with respect to the same modulus.*

The number of the numbers in (A) whose orders are m is denoted by $\phi(m)$ according to Gauss,* and this number is called the *Euler function* of m because Euler first found a general formula for its value. This number has also been called the *indicator* of m by Cauchy, and the *totient* of m by Sylvester. These three terms for the same function of m are still in common use. It is evident that $\phi(m)$ is the total number of the different sets, each set being composed of all congruent numbers (mod m), which involve the numbers of order m (mod m). When $m=p^{\alpha}$, p being a prime number, it is easy to see that

$$\phi(\mathrm{m})=\phi(p^{\alpha})=p^{\alpha}-p^{\alpha-1}=p^{\alpha-1}(p-1)=p^{\alpha}(1-1/p),$$

since only $p^{\alpha-1}$ of the numbers of (A) are divisible by p when $m=p^{\alpha}$, viz., the numbers,

$$p,\ 2p,\ 3p,\ \ldots,\ p^{\alpha-1}.p.$$

From this formula it results that exactly $p^{\alpha_{\beta}}_{\beta}(1-1/p_{\beta})$ of the given n_{β} numbers in (A) are of order $p^{\alpha_{\beta}}_{\beta}$, $1 \leqq \beta \leqq \lambda$. The complete system of residues (mod m)

$$n_1+n_2+n_3+\ldots+n_{\lambda} \tag{B}$$

must therefore contain exactly

Disquisitiones Arithmeticae, 1801, Art. 38; German edition, 1889, p. 22.

$$p_1^{\alpha_1}(1-\frac{1}{p_1}).p_2^{\alpha_2}(1-\frac{1}{p_2}).p_3^{\alpha_3}(1-\frac{1}{p_3})\dots p_\lambda^{\alpha_\lambda}(1-\frac{1}{p_\lambda})$$

$$=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})\dots(1-\frac{1}{p_\lambda})$$

numbers of order m, since the order of the sum of such a set of λ numbers is equal to the product of their orders according to the given theorem, and the choice of one of these λ numbers does not restrict the choice of another. We have therefore established Euler's formula,

$$\phi(m)=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})\dots(1-\frac{1}{p_\lambda}).$$

The number of numbers of order p_β^γ, $0 \overline{\leq} \gamma \overline{\leq} \alpha_\beta$, in n_β is $\phi(p_\beta^\gamma)$ since n_β contains exactly p_β^γ numbers whose orders divide p_β^γ and $1/p_\beta$ of these have orders which divide $p_\beta^{\gamma-1}$. Since all the numbers of (A) whose orders divide d are multiples of m/d it results that (A) contains exactly d numbers whose orders divide d if d is an arbitrary divisor of m. These d numbers include exactly $\phi(d)$ numbers of order d in accord with what was proved above. As all the numbers of (B) have an order which divides m it results from this that

$$m=\phi(d_1)+\phi(d_2)+\dots+\phi(d_l),$$

where d_1, d_2, ..., d_l are all the positive integral divisors of m, including 1, and m, and hence the second member of this equation gives the sum of the numbers of the integers of the same order in a complete system of residues (mod m).

If $m=m_1m_2$, where m_1, m_2 are relatively prime, we may assume that the orders of numbers in n_1, n_2, ..., n_δ divide m_1, while those in $n_{\delta+1}$, ..., n_λ divide m_2. Hence there are $\phi(m_1)$ numbers of order m_1 in $n_1+n_2+\dots+n_\delta$ and there are $\phi(m_2)$ numbers of order m_2 in $n_{\delta+1}+\dots+n_\lambda$. The sum of any one of the former $\phi(m_1)$ numbers and any one of the latter $\phi(m_2)$ numbers is of order m since these orders are relatively prime. Since this sum can be formed in $\phi(m_1).\phi(m_2)$ ways and the number of ways in which this sum can be formed is also $\phi(m)$, it results that $\phi(m)=\phi(m_1)\phi(m_2)$. This formula could also have been deduced directly from Euler's formula,

$$\phi(m)=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})\dots(1-\frac{1}{p_\lambda}).$$

§ 2. The Cyclic Group of Order m.

If an operation s has the period m it is said to generate the *cyclic group* of order m, and it may be represented by the different powers of s, as follows:

$$s,\ s^2,\ s^3,\ \ldots,\ s^m.$$

The exponents of s form the complete set of residues (mod m) which was denoted by (A) in the preceding section. As the combination* of two or more of these operations is effected by the addition of their exponents (mod m), it results that the abstract properties of the cyclic group are exhibited by the laws governing the addition (mod m) of the numbers in the set

$$1,\ 2,\ \ldots,\ m.$$

From this it follows directly that there is one and only one cyclic group of order m. From the fact that (A) contains exactly $p_\beta^{\alpha_\beta}$ numbers whose orders are powers of p_β, it results that the cyclic group of order m contains exactly $p_\beta^{\alpha_\beta}$ operators whose orders divide $p_\beta^{\alpha_\beta}$. The fact that the numbers

$$n_1+n_2+\ldots+n_\lambda \tag{B}$$

form a complete system of residues is equivalent to the fact that every element or operator or operation of a cyclic group is the product, in one and only one way, of elements whose orders are powers of prime numbers, no two of these orders being powers of the same prime.

The fact that there are exactly d numbers in (A) whose orders divide d, if d is an arbitrary divisor of m, is equivalent to the fact that a cyclic group contains one and only one subgroup whose order is an arbitrary divisor of the order of the group; and the formula

$$m=\phi(d_1)+\phi(d_2)+\ldots+\phi(d_l),$$

where $d_1,\ d_2,\ \ldots,\ d_l$ are all the different divisors of m, including 1 and m, merely asserts that if the numbers of all elements of the various different orders in a cyclic group are added together their sum is the order of this cyclic group.

The fact that $\phi(m)=\phi(m_1)\phi(m_2)$, if $m=m_1m_2$ and $m_1,\ m_2$ are relatively prime, is included in the theorem that a commutative group whose order is divisible by more than one prime is the product of its subgroups of

* This combination is commonly called finding the product of the operations involved.

relatively prime orders, and the order of the product of two commutative elements of a group is equal to the product of the order of these elements whenever these orders are relatively prime. The latter theorem is evidently a special case of the theorem, if the order of one of two commutative elements of a group is divisible by a higher power of a given prime than the order of the other then the order of the product of these two elements is also divisible by this higher power of the given prime, and the highest power of this prime which divides the order of the former element is also the highest power of this prime which divides the order of this product.

These parallel theorems may suffice to exhibit the fact that some of the most fundamental theorems in number theory are also fundamental theorems in group theory, and it seems unfortunate that our elementary books on number theory do not exhibit these points of contact more fully. It should be emphasized that the developments of §1 are not as simple as they would have been if the properties of the cyclic group had been first developed in the well known manner and if these had been employed in the proof of the given theorems.

The present mode of procedure has been adopted because of the fact that the cyclic group is so fundamental that it seems desirable to establish its fundamental properties in more than one way, and these propertios will doubtless appear more significant if they have been reached by different routes. It may also serve to illustrate relations between additive and multiplicative number theory. In fact, group theory owes a considerable part of its usefulness to the fact that it establishes close contact between additive and multiplicative number theory, as is fully illustrated by the preceding developments.

It may be added that the formula for the $\phi_r(m)$, viz.,

$$\phi_r(m)=m^r(1-\frac{1}{p_1{}^r})(1-\frac{1}{p_2{}^r})\dots(1-\frac{1}{p^r})$$

results directly from the fact that this formula gives the number of operators of order m in the direct product of r cyclic groups of order m, and hence also in the direct product of $r.\lambda$ cyclic groups of which there are r of each of the orders, $p_1{}^{a_1}$, $p_2{}^{a_2}$, ..., $p_\lambda^{a_\lambda}$. The number of operators of order $p_1{}^{a_1}$ in the direct product of the first r of these subgroups is clearly

$$p_1{}^{a_1r}-p_1{}^{a_1r-r}=p_1{}^{a_1r}(1-1/p_1{}^r),$$

and from this fact the given formula results immediately.*

* Cf. THE AMERICAN MATHEMATICAL MONTHLY, Vol. 11, (1904), p. 129.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

CALCULUS.

308. **Proposed by C. N. SCHMALL, New York City.**

Prove, by calculus, that of all isoperimetric triangles, the equilateral has the greatest area.

I. **Solution by J. SCHEFFER, A. M., Hagerstown, Md.**

Letting s represent the half sum of the three sides x, y, z, we have for the square of the area, $s(s-x)(s-y)(s-z)$; therefore $m=(s-x)(s-y)(s-z)=(s-x)(s-y)(x+y-s)=xy(x+y)-(x^2+3xy+y^2)s+2(x+y)s^2-s^3$ is to be a maximum. By differentiation

$$\frac{\partial m}{\partial x}=y(2x+y)-(2x+3y)s+2s^2,$$

$$\frac{\partial m}{\partial y}=x(2y+x)-(2y+3x)s+2s^2,$$

From $\dfrac{\partial m}{\partial x}=0$, $\dfrac{\partial m}{\partial y}=0$,
by subtracting one from the other, we get $(x-y)[s-(x+y)]=0$; whence $x-y=0$, the other root $x+y=s$ not being available. Hence $y=0$.

Substituting in either one, we have $3x^2-5xs+2s^2=0$; whence $x=s$, $x=\frac{2}{3}s$, the former of which is unavailable.

$\therefore x=\frac{2}{3}s$, $y=\frac{2}{3}s$, and $z=\frac{2}{3}s$.

II. **Solution by the PROPOSER.**

Let the given perimeter be $2s$; let the sides be x, y, z, and u the area. Then we have $z=2s-x-y$. Now

$$u=\sqrt{[s(s-x)(s-y)(x+y-s)]}\ldots(1),$$

which is to be rendered a maximum. Taking logarithms, we have

$$2\log u=\log s+\log(s-x)+\log(s-y)+\log(x+y-s).$$

$$\therefore \frac{2}{u}\frac{\partial u}{\partial x} = -\frac{1}{s-x} + \frac{1}{x+y-s}. \qquad \therefore \frac{\partial u}{\partial x} = \frac{u}{2}\frac{2s-2x-y}{(s-x)(x+y-s)} \ldots (2),$$

$$\text{and } \frac{\partial u}{\partial y} = \frac{u}{2}\frac{2s-2y-x}{(s-y)(x+y-s)} \ldots (3).$$

Equating (2) and (3) to 0, we have $2s-2x-y=0$, $2s-2y-x=0$, whence $x=\frac{2}{3}s$, $y=\frac{2}{3}s$, $z=2s-x-y=\frac{2}{3}s$.

Hence the triangle is equilateral; and there is evidently a maximum.

309. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology.

In practical problems involving maxima and minima, it is really the greatest and least values of the function which are desired. Show why we can assume that the maximum is the greatest value and the minimum the least value under the conditions.

No solution of this problem has been received.

310. Proposed by C. N. SCHMALL, New York City.

Evaluate $\int_0^{\pi} \frac{dx}{1-2a\cos x+a^2}$. Edwards' *Integral Calculus for Beginners*, page 131, ex. 9, (iii). The answer given is $\frac{\pi}{1-a^2}$. Is this a complete answer to the question?

I. Solution by H. PRIME, Boston, Mass.

$$\int_0^{\pi} \frac{dx}{1+a^2-2a\cos x} = \left[\frac{2}{\pm(1-a^2)}\tan^{-1}\left(\pm\frac{1+a}{1-a}\tan\frac{x}{2}\right)\right]_0^{\pi}$$

$$= \frac{(2n+1)\pi}{\pm(1-a^2)} - \frac{2n\pi}{\pm(1-a^2)} = \frac{\pi}{\pm(1-a^2)}.$$

It should be noted that if $a=\pm1$, the function $1/(1+a^2-2a\cos x)$ becomes infinite for $x=0$ or $x=\pi$. Hence, in this case, the integral has no meaning for the required limits, as they do not exist. The result also shows this for $\pi \div (1-a^2)=\infty$, when $a=\pm1$.

Also solved by J. Scheffer, Francis Rust, A. M. Harding, and V. M. Spunar.

311. Proposed by WILMER THOMPSON, Senior, Drury College.

Solve the differential equation,

$$\left(\frac{dy}{dx}\right)^3 + x^3 = ax\left(\frac{dy}{dx}\right).$$

[From Forsythe's *Differential Equations*, p. 47.]

I. Solution by W. W. BEMAN, Professor of Mathematics, University of Michigan, and A. H. HOLMES, Brunswick, Maine.

$p^3+x^3=axp$. Putting $p=xu$, $x=\dfrac{au}{1+u^3}$, $p=\dfrac{au^2}{1+u^3}$.

Further, $p=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=\dfrac{(1+u^3)^2}{a\,(1-2u^3)}\cdot\dfrac{dy}{du}$.

$\therefore\ \dfrac{dy}{du}=a^2\dfrac{(1-2u^3)u^2}{(1+u^3)^3}$. Integrating, $y+c=\dfrac{a^2}{6}\cdot\dfrac{1+4u^3}{(1+u^3)^2}$.

Eliminating u from this equation and $x=\dfrac{au}{1+u^3}$,

$$216(y+c)^3-36a^2(y+c)^2-108ax^3(y+c)+16a^3x^3+27x^6=0.$$

II. Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Expressing the above equation in the form, $p^3-axp+x^3=0$, and solving for p by Cardan's Method, we obtain

$$p=\left(-\frac{x^2}{2}+\sqrt{+\frac{x^6}{4}-\frac{a^3\,x^3}{27}}\right)^{1/3}+\left(-\frac{x^3}{2}-\sqrt{+\frac{x^6}{4}-\frac{a^3\,x^3}{27}}\right)^{1/3}\ldots(1).$$

Assuming $-\dfrac{a^3\,x^3}{27}=-\dfrac{x^6}{4}\sin^2\theta\ldots(2)$, (1) becomes

$$p=(-x)\left[(\cos\tfrac{1}{2}\theta)^{2/3}+(\sin\theta)^{2/3}\right]\ldots(3),\ \text{ and }\ x=\frac{4^{1/3}a}{3(\sin\theta)^{2/3}}\ldots(4).$$

From (4), $dx=-\dfrac{2.4^{1/3}a}{9}\dfrac{\cos\theta}{(\sin\theta)^{\frac{5}{3}}}d\theta$. By substituting in (1),

$$p=\frac{2.16^{2/3}\,a^2}{27}\left[\frac{(\cos\frac{1}{2}\theta)^{2/3}}{(2\sin\frac{1}{2}\theta\,\cos\frac{1}{2}\theta)^{2/3}}+\frac{(\sin\frac{1}{2}\theta)^{2/3}}{(2\sin\frac{1}{2}\theta\,\cos\frac{1}{2}\theta)^{2/3}}\right]\frac{(2\cos^2\frac{1}{2}\theta-1)}{(2\sin\frac{1}{2}\theta\,\cos\frac{1}{2}\theta)^{\frac{5}{3}}}$$

$$dy=\frac{2.16^{1/3}.a^2}{27.2^{\frac{7}{3}}}\left[\frac{2(\cos\frac{1}{2}\theta)^{1/3}}{(\sin\frac{1}{2}\theta)^{\frac{7}{3}}}-\frac{1}{(\sin\frac{1}{2}\theta)^{\frac{7}{3}}\,(\cos\frac{1}{2}\theta)^{\frac{5}{3}}}+\frac{2}{(\sin\frac{1}{2}\theta)^{\frac{5}{3}}\,(\cos\frac{1}{2}\theta)^{1/3}}\right.$$

$$\left.-\frac{1}{(\sin\frac{1}{2}\theta)^{\frac{5}{3}}\,(\cos\frac{1}{2}\theta)^{\frac{7}{3}}}\right]d\theta.$$

Placing $\tan\theta=t$, $d\theta=\dfrac{dt}{1+t^2}$, $\sin\theta=\dfrac{t}{(1+t^2)^{½}}$, and $\cos\theta=\dfrac{1}{(1+t^2)^{½}}$.

Integrating, we obtain,

$$y+c=-\tfrac{3}{4}\left[(\tan\tfrac{1}{2}\theta)^{-\frac{4}{3}}+(\tan\tfrac{1}{2}\theta)^{\frac{4}{3}}\right]\frac{2.16^{⅓}a^2}{27.2^{\frac{7}{3}}}=-\frac{a^2}{36}\left[\left(\tan\tfrac{1}{2}\theta\right)^{-\frac{4}{3}}+\left(\tan\tfrac{1}{2}\theta\right)^{\frac{4}{3}}\right]$$

from which, by substitution, the solution may be completed.

III. Solution by WILLIAM HOOVER, Athens, Ohio.

Let $dy/dx=p$, as usual; then $p^3-axp+x^3=0$.
The condition for equal values of p from this cubic is

$$\frac{p^6}{4}=\frac{a^3p^3}{27},\text{ or } p=\frac{dy}{dx}=\frac{a}{3}\sqrt[3]{4}.$$

Integrating, $\dfrac{3y}{a\sqrt[3]{4}}=x+c$, or, $27y^3=4a^3(x+c)^3$.

Also solved by S. G. Barton.

MECHANICS.

255. Proposed by the late G. B. M. ZERR, Ph. D.

Assuming the resilience of volume of mercury to be constant at all depths and to be 54.20×10^{10} in *C. G. S.* units and that a mile = 160933 centimeters. Find the depth of an ocean of mercury at a point where its density is double the surface density, 13.596.

Solution by B. F. FINKEL, Ph. D., Drury College.

Let x_0 be the length of a column of mercury of one square centimeter cross-section and uncompressed, such that its weight is sufficient to compress a cubic centimeter to half its volume, which is equivalent to doubling its density.

By Hooke's law, $p=E\dfrac{\triangle v}{v}=\tfrac{1}{2}E$, where E is the coefficient of elasticity. But $p=13.596gx_0$.

$\therefore x_0=\dfrac{E}{27.192g}$. Let $\triangle x_0$=the amount of compression of the column by virtue of its own weight, and let x=the length of any portion of the column measured from the surface downward, and let dx be an element of length of this column. Then the strain, s, in this element is

$s=\dfrac{d(\triangle x_0)}{dx}$. But by Hooke's Law, $s=\dfrac{p}{E}$. Hence, $\dfrac{d(\triangle x_0)}{dx}=\dfrac{p}{E}$

$=\dfrac{13.596gx}{E}$, or $d(\triangle x_0)=\dfrac{13.596gxdx}{E}$, and

$$\therefore \triangle x_0=\frac{13.596g}{E}\int_0^{x_0} xdx=\frac{13.596g(\frac{1}{2}x_0^2)}{E}=\frac{6.798gx_0^2}{E}$$

$\therefore$ The depth of the ocean required is $x=x_0-\triangle x_0$

$$=\frac{E}{27.192g}-\frac{6.798g}{E}\left(\frac{E}{27.192g}\right)^2=\frac{E}{27.192g}\left[1-\tfrac{1}{4}\right]$$

$$=\frac{3E}{108.768g}\text{cm.}=\frac{3E}{108.768\times160933\times g}\text{mi.}$$

Using $g=981$, and $E=33\times10^{10}$, we have $x=61.8$ miles.

REMARK. We are unable to get data from which to obtain Dr. Zerr's value for the volume resilience. The value of E as given in Kimball's *Physics*, page 158, and which we have used in the above solution, is very different from what would be obtained from Dr. Zerr's value for volume resilience.

NOTES AND NEWS.

The fifth regular meeting of the Southwestern Section of the American Mathematical Society was held at Washington University, St. Louis, Missouri, on Saturday, December 2, 1911. S.

The eighteenth summer meeting of the American Mathematical Society was held at Poughkeepsie, New York, on September 12-13, 1911. There were about thirty-five members in attendance and twenty-six papers were presented. S.

The twenty-ninth regular meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago on Friday and Saturday, December 29, 30, 1911. Titles and abstracts of papers to be presented at this meeting should be sent to the secretary of the Section, Professor H. E. Slaught, the University of Chicago, Chicago, Illinois. S.

Dr. Jacques Hadamard, Professor of Analytic and Celestial Mechanics in the Collége de France, lectured at Columbia University five times each week in October. He also delivered two lectures at the University of Chicago on October 31 on the topics: (1) Certain Mathematical Improvements likely to be useful in the study of Physics; (2) Psychology of Mathematicians. S.

We learn from *Science* that the Jean Reynaud prize of ten thousand francs, awarded by the Paris Academy of Sciences every five years, has been bestowed this year on Professor Emile Picard, for his contributions to mathematics. And from the same journal we learn that the DeMorgan medal of the London Mathematical Society has been awarded to Professor Horace Lamb, F. R. S., for his research in Mathematical Physics. F.

Editor Miller, in *Science* for December 1, has pointed out a number of mathematical errors in the eleventh edition of *Encyclopedia Britannica.* Among the errors pointed out by Professor Miller are, (1) the incorrect statement of the origin of *zero*, (2) the first use of the term *abscissa*, and (3) the first use of $\phi(n)$. We remark in passing that in our judgment the eleventh edition of this monumental work is not up to the standard of excellence for the mathematician as is the ninth edition. F.

The annual conference of the University of Chicago with the secondary schools, which has heretofore been held in November, has been postponed till March, 1912, in order that a comprehensive plan of visitations may be carried on during the autumn and winter, both by secondary teachers at the University and by instructors in the University among the schools. The reports of these visitations are to form the basis of the next conference and it is expected that much mutual benefit will be derived from such a scheme of co-operation between the schools and the University. S.

Several of the reports of the American committees of the International Commission on the Teaching of Mathematics have been published by the United States Bureau of Education, and are not being distributed by the bureau. These reports are of the greatest value to teachers of mathematics and should be eagerly sought by them.

There has come to us the report of committees Nos. V, VII, IX, X, XII. Committee No. V made a report on the Training of Teachers of Elementary and Secondary Mathematics; Committee No. VII, on Examinations in Mathematics Other Than Those Set by the Teacher for His Own Classes; Committee No. IX, Mathematics in the Technological Schools of Collegiate Grade in the United States; committee No. X, Undergraduate Work in Mathematics in Colleges of Liberal Arts and Universities; and Committee No. XII, Graduate Work in Mathematics in Universities and in Other Institutions of Like Grade in the United States. F.

The one hundred and fifty-fifth regular meeting of the American Mathematical Society was held at Columbia University, New York City, on Saturday, October 28, 1911. S.

The preliminary report of the National Committee of fifteen on Geometry Syllabus was the subject of discussion at the autumn meeting of the Middle States and Maryland Association of teachers of mathematics. It was also discussed at the meeting of the mathematics section of the high school teachers of Chicago, November 11, 1911. There is a constant demand for copies of this report which cannot be supplied until the committee complete the revision at which they are now working, when another edition will be printed for general distribution and a regular sale price will be determined. S.

One of the editors desires to put on record a dream in regard to mathematical notation. He thought he was visiting a class in elementary mathematics and heard the instructor say "x sub sucker r from 0 to 10." Several students inquired what he meant and he explained that this was his way of expressing the relation x_r, $0 \overset{=}{<} r \overset{=}{<} 10$, as r was between the suckers $<$, $<$. Moreover, this appeared to be in accord with the statement in the Scriptures, "Whosoever hath, to him shall be given, and he shall have more abundance; but whosoever hath not, from him shall be taken away even that he hath." The symbol $<$ seemed to suck from the smaller towards the larger. M.

The last annual meeting of the Central Association of Science and Mathematic Teachers was held at Lewis Institute, Chicago, Illinois, on November 30 and December 1, 1911. This association has for years been carrying forward progressive and consistent work along all lines of science and mathematics. In the latter it has developed reports on algebra, geometry, and unified mathematics, which have had widespread influence all over the country, but especially in the middle West. Every secondary teacher of any of these subjects within this territory should be allied with this association. *School Science and Mathematics* is the official publication of this association. S.

The annual conference of the University of Illinois with the secondary schools was held on Thursday, Friday, Saturday, November 23-25, 1911. An important report of a committee on Geometry was made at this meeting. The University of Illinois has conducted for several years, in co-operation with the high schools, a careful study of the secondary curriculum, with a view to formulating programs of study in the various subjects. Two years ago an excellent report on Algebra was adopted and incorporated in the official publication of the conference. Last year the Geometry report in preliminary form was the topic in two long sessions of the Mathematical Section. This year it was presented in final form. S.

BOOKS.

The Teaching of High School Mathematics, by George W. Evans, Headmaster of the Charlestown High School, one of the Riverside Educational Monographs, edited by Henry Suzzallo, Professor of the Philosophy of Education, Teachers College, Columbia University: 8vo. Cloth, viii+94 pages. Price, 35 cents. Boston: Houghton, Mifflin & Co.

In addition to the Editor's Introduction, the book contains a brief discussion of the following: The Modern Point of View; The Order of Topics; Equations and their Use; Some Rules of Thumb; Geometry as Algebraic Material; The Graphical Method; The Bases of Proof in Geometry; The Method of Limits; Simpson's Rule and the Curve of Sections; The Teacher. The Author presents some very helpful and suggestive thoughts on elementary teaching and his book is worthy a careful study by the teacher of elementary mathematics. F.

A College Text-Book of Physics. By Arthur L. Kimball, Ph. D., Professor of Physics in Amherst College. 8vo. Cloth, ix+692 pages. Price, $2.75. New York: Henry Holt & Co.

This is one ofthe most illuminating texts on physics for college use that has appeared in recent years. The method of presentation and the clearness of exposition is very attractive and will appeal strongly to those teachers who want a good text with not too much mathematical development of physical principles. F.

Plain and Solid Geometry with Problems and Applications. By H. E. Slaught, Ph. D., Associate Professor of Mathematics in the University of Chicago and N. J. Lennes, Ph. D., Instructor in Mathematics in Columbia University. 8vo. Cloth, xii+470 pages. Boston: Allyn & Bacon.

In writing this book, the authors were guided by the following two purposes, viz: (a) that pupils may gain by gradual processes the power and habit of deductive reasoning and (b) that they may learn to know the facts of elementary geometry as elementary properties of the space in which they live.

They have made use freely of the best of the many suggestions offered by various committees appointed at various times during the past five or six years, for the improvement of the teaching of geometry. Many problems of very practical applications have been incorporated in the work, and it is believed that these will stimulate an interest in the minds of most students who use the book. The book has received the most hearty indorsement by the educational public, a fact evinced by the many schools in which it was adopted immediately after coming from the press. F.

Vector Analysis. An introduction to Vector-Methods and their various applications to Physics and Mathematics. By Joseph George Coffin, B. S., Ph. D., (Massachusetts Institute of Technology, '98, and Clark University, '03) Instructor in Physics at the College at the City of New York. Second edition. 12 mo. Cloth, xviii+262 pages. Price, $2.50. New York: John Wiley & Sons.

The cordial reception of the first edition of this book both by American and European Mathematicians has encouraged the author to bring out this second edition. Certain portions of the work have been rewritten and 14 pages of notes have been added to the appendix. The book is doing great good in leading teachers of physics to an understanding of the use of a very powerful instrument for physical research. F.

An Introduction to Mathematics,. By A. N. Whitehead, S. C. D., F. R. S. Author of "Universal Algebra." 12mo. Cloth, 256 pages. Price, 75 cents. New York: Henry Holt & Co.

"Mr. Whitehead sets out not to teach mathematics, but to enable students from the very beginning of their course to know what the science is about and why it is necessarily the foundation of exact thought as applied to natural phenomena? It is just because mathematical ideas are abstract that they supply what is wanted for scientific description of the course of events, freed from reference to particular persons or particular types of sensation. From this starting-point the author proceeds to explain the true inwardness of 'variables,' 'dynamics,' the symbolism of mathematics, generalizations of numbers, imaginary numbers, co-ordinate geometry, conic sections, functions, periodicity, trigonometry, the differential calculus, and geometry. An admirably clear exposition, illustrated throughout with diagrams."

Analytical Mechanics, Comprising the Kinetics and Statics of Solids and Fluids. By Edwin H. Barton, D. Sc. (Lond.), F. R. S. E., F. Ph. S. L., Professor of Experimental Physics, University College, Nottingham. 8vo. Cloth, xx+535 pages, with diagrams. London: Longmans, Green & Co.

This book, which requires on the part of the student an elementary knowledge of the calculus, gives a fairly complete treatment of the kinetics and statics of solids and of fluids, closing with a brief chapter on elasticity.

The author, after a brief introduction, takes up the treatment of mechanics under the following: kinematics, kinetics, statics, hydromechanics, and elasticity, and throughout the body of the work are scattered at frequent intervals sets of examples; while at the end is a list of miscellaneous problems of various degrees of difficulty.

The treatment of the various subjects is very clear and direct and well adapted to the needs of the students of engineering. F.

Physics for College Students. By Henry S. Carhart, LL. D., Professor Emeritus of Physics, University of Michigan. 8vo. Cloth, viii+622 pages. Boston: Allyn & Bacon.

The author informs us that this book was prepared in response to many inquiries for a text somewhat more advanced than his High School Physics and distinctively less mathematical than his *University Physics*. We believe that Professor Carhart has succeeded in preparing a book at once scientific and teachable and that the treatment here presented will lead to satisfactory results. F.

College Physics. By John Oren Reed, Ph. D., Professor of Physics in the University of Michigan and Dean of the Department of Literature, Science, and the Arts and Karl Eugene Guthe, Ph. D., Professor of Physics in the University of Michigan. 8vo. Coth, xxviii+622 pages. New York: Macmillan & Co.

This text is somewhat similar in treatment to that of the Kimball Text, and in the hands of a good teacher will produce first class results. The authors in the preparation of the book kept three things in mind, we are told: First, to present the fundamental facts of physics in a clear, concise and teachable form; second, to relate these fundamental facts to the basic laws and to the theories of physics in such way as to render plain the historical growth of the science; and, third, to put the student in direct touch with first hand information concerning the epoch-making discoveries of the past upon which the growth of the science has been based, as well as to afford an intimation of the marvelous progress of the present. F.

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AVERAGE AND PROBABILITY.

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NOTE ON THE DERIVATIVE OF THE QUOTIENT OF TWO WRONSKIANS.

By L. A. HOWLAND, Middletown, Connecticut.

Suppose we have given a set of $n+1$ analytic functions of x, in which there are two sets of n functions each, which are linearly independent. A formula for the derivative of the quotient of the Wronskians of these two sets has important application in the theory of linear differential equations and elsewhere.* The formula is supposed to be due to Frobenius, who derived and applied it in the article cited. It may be of interest to note that we can reverse the application and use the theory of linear differential equations to establish the formula.

Since it is merely a question of notation, we shall assume the sets to be: (1) the first n functions, and (2) the first $n-1$ and the last. We have then to evaluate

$$\text{(1)} \qquad \frac{d}{dx}\,\frac{W_2(a_1,\ a_2,\ \ldots,\ a_{n-1},\ a_{n+1})}{W_1(a_1,\ a_2,\ \ldots,\ a_{n-1},\ a_n)}=\frac{W_1W_2'-W_1'W_2}{W_1^2}$$

where $W_1=W_1(a_1,\ a_2,\ \ldots,\ a_n)=\begin{vmatrix} a_1 & a_2 & \ldots & a_n \\ a_1' & a_2' & \ldots & a_n' \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{(n-1)} & a_1^{(n-1)} & \ldots & a_n^{(n-1)} \end{vmatrix}$

and similarly for W_2.

Form the equation:

$$\text{(2)} \qquad W_1W_2'-W_1'W_2-W(a_1,\ \ldots,\ a_{n-1})\,W(a_1,\ \ldots,\ a_{n+1})=0,$$

or written out in determinant form:

* Cf., for example, Frobenius, Crelle, 77, p. 248; Schlesinger, Handbuch d. Theorie d. Lin. Diff. Gl., Bd. 1, p. 60; Curtiss, Math. Ann., Bd. 65, p. 284; also Bull. Am. Math. Soc., Vol. XVII, p. 465.

$$\begin{vmatrix} a_1 & \dots & a_n \\ a_1' & \dots & a_n' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_1^{(n-1)} & \dots & a_n^{(n-1)} \end{vmatrix} \begin{vmatrix} a_1 & \dots & a_{n-1} & a_{n+1} \\ a_1' & \dots & a'_{n-1} & a'_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{(n-2)} & \dots & a_{n-1}^{(n-2)} & a_{n+1}^{(n-2)} \\ a_1^{(n)} & \dots & a_{n-1}^{(n)} & a_{n+1}^{(n)} \end{vmatrix} - \begin{vmatrix} a_1 & \dots & a_n \\ a_1' & \dots & a_n' \\ \cdot & \cdot & \cdot \\ a_1^{(n-2)} & \dots & a_n^{(n-2)} \\ a_1^{(n)} & \dots & a_n^{(n)} \end{vmatrix}$$

$$\begin{vmatrix} a_1 & \dots & a_{n-1} & a_{n+1} \\ a_1' & \dots & a'_{n-1} & a'_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{(n-1)} & \dots & a_{n-1}^{(n-1)} & a_{n+1}^{(n-1)} \end{vmatrix} - \begin{vmatrix} a_1 & \dots & a_{n-1} \\ a_1' & \dots & a'_{n-2} \\ \cdot & \cdot & \cdot \\ a_1^{(n-2)} & \dots & a_{n-1}^{(n-2)} \end{vmatrix} \begin{vmatrix} a_1 & \dots & a_{n+1} \\ a_1' & \dots & a'_{n+1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_1^{(n-1)} & \dots & a_{n+1}^{(n-1)} \\ a_1^{(n)} & \dots & a_{n+1}^{(n)} \end{vmatrix} = 0.$$

This is obviously a homogeneous linear differential equation in a_{n+1} of order at most equal to n. Its order, however, is seen by inspection to be less than n, for the coefficient of $a_{n+1}^{(n)}$ in the expansion is

$$W(a_1 \dots a_n)\, W(a_1 \dots a_{n-1}) - W(a_1 \dots a_{n-1})\, W(a_1 \dots a_n) \equiv 0.$$

Again by inspection it is easily seen that the equation has the n integrals a_1, $a_2 \dots a_n$, for the second determinant in each term has two columns alike when a_{n+1} is put equal to a_1, $a_2 \dots a_{n-1}$, whereas when $a_{n+1} = a_n$, the first two terms become identical and the last vanishes as before. But a linear equation of this sort and of order less than n which has n linearly independent integrals vanishes identically and hence (2) is an identity.

We have thus the formula:

$$\frac{d}{dx} \frac{W_2(a_1 \dots a_{n-1}\, a_{n+1})}{W_1(a_1 \dots a_{n-1}\, a_n)} = \frac{W(a_1 \dots a_{n-1})\, W(a_1 \dots a_{n+1})}{[W(a_1 \dots a_n)]^2}.$$

Finally, we will note concerning this proof, as did Professor Curtiss concerning the derivation of the formula by Frobenius,* that although we started with analytic functions, the formula holds for any functions having the necessary derivatives. In fact, formula (2) holds when the primes, etc., lose the significance of differentiation, for the vanishing of the first member of (2) is purely formal; by that we mean that it vanishes identically when regarded as a polynomial in the $(n+1)^2$ independent variables $a_1 \dots a_{n+1}$, $a_1' \dots a'_{n+1} \dots a_{n+1}^{(n)}$, since the first $n+1$ of these, the original functions, may be so chosen that each of these symbols has at a given point any prescribed value. As an example:

---

* Math. Ann., Bd. 65, foot-note on p. 284.

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & a_{41} \\ a_{12} & a_{22} & a_{42} \\ a_{14} & a_{24} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{14} & a_{24} & a_{34} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & a_{41} \\ a_{12} & a_{22} & a_{42} \\ a_{13} & a_{23} & a_{43} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix}$$

ON A SPECIAL CASE OF EQUILIBRIUM OF A FLEXIBLE, INEXTENSIBLE STRING.

By CHARLES C. STECK, Durham, New Hampshire.

It is proposed in this paper to discuss the curve of equilibrium of a weightless, flexible, inextensible string, the end points of which are attached to two fixed points, each element of the string being acted upon by a force always normal to an axis which it meets, this force being any function of the distance r of the element from the given axis. A special discussion is made of the case where the force varies inversely as r^2.

The general problem as here stated is suggested by Appell in his *Traité de Mécanique Rationelle*, Vol. I, Chap. VII. In connection with this problem he gives the three following equations which are directly obtainable from the results he has arrived at in his discussion of equilibrium of strings:

$$(1) \quad T\frac{dz}{ds}=c; \quad (2) \quad Tr^2\frac{d\theta}{ds}=k; \quad (3) \quad T=-\int F(r)\,dr-h=\phi(r),$$

in which T is the tension at any point, $F(r)$ the force, the z-axis the axis to which the force is normal, r and θ the polar co-ordinates in the xy plane, and c, k, h arbitrary constants.

Starting with these three equations and with the equation

$$(4) \qquad ds^2=dr^2+r^2d\theta^2+dz^2,$$

we have, on eliminating T,

$$(5) \qquad k\frac{dr}{rd\theta}=\pm\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}},$$

(6) $$cr\frac{dr}{dz}=\pm\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}},$$

(7) $$r\phi(r)\frac{dr}{dz}=\pm\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}}.$$

On integrating these equations we obtain

(8) $$\theta-\theta_0=\pm k\int\frac{dr}{r\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}}},$$

(9) $$z-z_0=\pm c\int\frac{rdr}{\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}}},$$

(10) $$s-s_0=\pm\int\frac{r\phi(r)\,dr}{\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}}}.$$

Equations (8) and (9) define the curve of equilibrium of the string. Equation (8) is the equation of a cylindrical surface the elements of which are perpendicular to the plane xy. Equation (9) is the equation of a surface of revolution, the axis of revolution being the z-axis. Equations (8), (9), and (10) contain six arbitrary constants which are to be determined from initial conditions.

Special Cases.

Case I. $c=0$, $k\neq0$.

From (1) when $c=0$, either $T=0$ or $\frac{dz}{ds}=0$. Since T cannot be zero when $F(r)$ is not zero, $\frac{dz}{ds}$ must be zero; that is, the curve of equilibrium is in a plane perpendicular to the z-axis. Equations (8), (9), and (10) take the following forms:

(11) $$\theta-\theta_0=\pm k\int\frac{dr}{r\sqrt{\{[\phi(r)]^2r^2-k^2\}}},$$

(12) $$z-z_0=0,$$

(13) $$s-s_0=\pm\int\frac{r\phi(r)\,dr}{\sqrt{\{[\phi(r)]^2r^2-k^2\}}}.$$

Equations (11) and (12) define the curve of equilibrium of a string under the action of a central force.

Case II. $k=0$, $c\neq 0$, $r\neq 0$.

From (2), when $k=0$, either $T=0$ or $r^2\dfrac{d\theta}{ds}=0$. As in Case I, T cannot be zero, and it follows that $r^2\dfrac{d\theta}{ds}$ must be zero; that is, $\dfrac{d\theta}{ds}=0$ or $\theta=\theta_0$. The curve of equilibrium lies in a plane perpendicular to the xy plane and passing through the z-axis. Equations (8), (9), and (10) take the forms

$$\theta-\theta_0=0, \tag{14}$$

$$z-z_0=\pm c\int\frac{dr}{\sqrt{\{[\phi(r)]^2-c^2\}}}, \tag{15}$$

$$s-s_0=\pm\int\frac{\phi(r)\,dr}{\sqrt{\{[\phi(r)]^2-c^2\}}}. \tag{16}$$

Case III. $k=0$, $c=0$, $r\neq 0$.

For this case equations (8), (9), and (10) take the forms,

$$\theta-\theta_0=0, \tag{17}$$

$$z-z_0=0, \tag{18}$$

$$s-s_0=r. \tag{19}$$

The curve of equilibrium is a straight line perpendicular to the z-axis. Since this is true whatever be the force, so long as it is a function of r and r is nowhere zero, it will be unnecessary to consider this case in the discussion which follows and which deals with a special force.

Discussion of the Case Where $F(r)=\frac{\mu}{r^2}$, $\mu>0$.

Since we shall deal only with finite forces, it follows that r can nowhere be zero; that is, the string cannot touch the z-axis at any point. Further, since $\mu>0$, we shall have a repulsive force. This constant μ is the value of the force at a unit's distance from the z-axis. Substituting the above value of $F(r)$ in equation (3) and integrating, we have

$$\phi(r)=\frac{\mu}{r}-h=T. \tag{20}$$

Substituting this value of $\phi(r)$ in equations (8), (9), and (10), we get,

$$\theta-\theta_0=\pm k\int\frac{dr}{r\sqrt{\{r^2(h^2-c^2)-2h\mu r+\mu^2-k^2\}}}, \tag{21}$$

$$z-z_0=\pm c\int\frac{rdr}{\sqrt{\{r^2(h^2-c^2)-2h\mu r+\mu^2-k^2\}}}, \tag{22}$$

$$s-s_0=\pm\int\frac{(\mu-hr)dr}{\sqrt{\{r^2(h^2-c^2)-2h\mu r+\mu^2-k^2\}}}. \tag{23}$$

These three indicated integrations can be immediately performed. The following cases must be distinguished.

I. $h^2>c^2$, $\mu^2>k^2$.
 (a) $c^2\neq0$, $k^2\neq0$.
 (b) $c^2=0$, $k^2\neq0$.
 (c) $c^2\neq0$, $k^2=0$.
II. $h^2>c^2$, $\mu^2=k^2$.
 (a) $c^2\neq0$.
 (b) $c^2=0$.
III. $h^2>c^2$, $\mu^2<k^2$.
 (a) $c^2\neq0$.
 (b) $c^2=0$.
IV. $h^2<c^2$, $\mu^2>k^2$.
 (a) $h^2\neq0$, $k^2\neq0$.
 (b) $h^2=0$, $k^2\neq0$.
 (c) $h^2\neq0$, $k^2=0$.
 (d) $h^2=0$, $k^2=0$.
V. $h^2<c^2$, $\mu^2=k^2$.
 (a) $h^2\neq0$.
 (b) $h^2=0$.
VI. $h^2<c^2$, $\mu^2<k^2$.
 (a) $h^2\neq0$.
 (b) $h^2=0$.
VII. $h^2=c^2$, $\mu^2>k^2$.
 (a) $h^2\neq0$, $k^2\neq0$.
 (b) $h^2=0$, $k^2\neq0$.
 (c) $h^2\neq0$, $k^2=0$.
VIII. $h^2=c^2$, $\mu^2=k^2$.
 (a) $h^2\neq0$.
 (b) $h^2=0$.
IX. $h^2=c^2$, $\mu^2<k^2$.
 (a) $h^2\neq0$.
 (b) $h^2=0$.

The following notations are used:

$$\begin{aligned}
&R\equiv r^2(h^2-c^2)-2h\mu r+\mu^2-k^2,\\
&m^2\equiv h^2-c^2 \text{ if } h^2>c^2,\\
&-m^2\equiv h^2-c^2 \text{ if } h^2<c^2,\\
&n^2\equiv\mu^2-k^2 \text{ if } \mu^2>k^2,\\
&-n^2\equiv\mu^2-k^2 \text{ if } \mu^2<k^2,\\
&\Delta\equiv h^2\mu^2-(h^2-c^2)(\mu^2-k^2).
\end{aligned}$$

We give here an outline of the method used in determining the constants for Case I (a). The results for this case and for all of the others are put in tabular form at the end, followed by a series of notes by which necessary explanations are made.

DETERMINATION OF THE CONSTANTS FOR CASE I (a).

Initial conditions. Let the initial value of r be that value which makes $R=0$. Call this value r_1. When $r=r_1$, let $\theta=0$, $z=0$, $s=0$, $T=T_1$.

Interpretation of initial conditions. Returning to equations (1) and (2), we see that when $R=0$, since $k\neq 0$ and $c\neq 0$, $\frac{1}{r}\frac{dr}{d\theta}=0$ and $\frac{dr}{dz}=0$. Now $\frac{1}{r}\frac{dr}{d\theta}$ is the cotangent of the angle between the radius vector and the tangent to the guiding curve of the cylinder at any point. We shall call this angle ϕ. $\frac{dr}{dz}$ gives the slope of the tangent to the meridian section of the surface of revolution at the same point. Call this angle ψ. With the initial conditions just imposed $\phi_1=\frac{1}{2}\pi$ and $\psi_1=0$.

From (1) it is evident that c is the value of the tension when $\frac{dz}{ds}=1$; *i. e.*, the component of the tension parallel to the z-axis; call it T_0. c may always be taken positively if we measure s in such a direction that z and s increase simultaneously since T is essentially positive. The constant h may be found from equation (20). We have, on imposing initial conditions

$$h=\frac{\mu}{r_1}-T_1.$$

The constant k is found from the equation $R=0$. Solving this equation for k^2 we get

$$k^2=r_1^2(T_1^2-T_0^2).$$

The values of θ_0, z_0, and s_0 are

$$\theta_0=0,\ z_0=0,\ s_0=0.$$

The corresponding values of m^2, n^2, and Δ are

$$m^2=h^2-c^2=\frac{1}{r_1^2}[\mu^2-2\mu r_1T_1+r_1^2(T_1^2-T_0^2)],$$

$$n^2=\mu^2-k^2=\mu^2-r_1^2(T_1^2-T_0^2),$$

$$\Delta=[\mu T_1-r_1(T_1^2-T_0^2)]^2.$$

Case			r_1	Equation of Cylinder	θ_0	Equation of Surface of Revolu.	z_0	Equation of Length of Arc	s_0	Notes
I $h^2>c^2$ $\mu^2>k^2$	a	$c^2\neq0$ $k^2\neq0$	$\frac{h\mu-\sqrt{\Delta}}{m^2}$	$r=\frac{n^2}{h\mu+\sqrt{\Delta}\cosh\left[\pm\frac{n}{k}(\theta-\theta_0)\right]}$	0	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\cosh^{-1}\left(\frac{h\mu-m^2r}{\sqrt{\Delta}}\right)\right\}$	0	$s-s_0=\pm\left\{\frac{\mu}{m}\left(1-\frac{h^2}{m^2}\right)\cosh^{-1}\left(\frac{h\mu-m^2r}{\sqrt{\Delta}}\right)+\frac{h\sqrt{R}}{m^2}\right\}$	0	A B
	b	$c^2=0$ $k^2\neq0$	$\frac{h\mu-\sqrt{\Delta}}{m^2}$	$r=\frac{n^2}{h\mu+\sqrt{\Delta}\cosh\left[\pm\frac{m}{k}(\theta-\theta_0)\right]}$	0	$z-z_0=0$	0	$s-s_0=\mp\frac{\sqrt{R}}{h}$	0	A B
	c	$c^2\neq0$ $k^2=0$	$\frac{h\mu-\sqrt{\Delta}}{m^2}$	$\theta-\theta_0=0$	0	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\cosh^{-1}\left(\frac{h\mu-m^2r}{\sqrt{\Delta}}\right)\right\}$	0	$s-s_0=\pm\left\{\frac{\mu}{m}\left(1-\frac{h^2}{m^2}\right)\cosh^{-1}\left(\frac{h\mu-m^2r}{\sqrt{\Delta}}\right)-\frac{h\sqrt{R}}{m^2}\right\}$	0	A B
II $h^2>c^2$ $\mu^2=k^2$	a	$c^2\neq0$	$\frac{k\cot\phi_1}{c\tan\psi_1}$	$r=\frac{2\mu h}{m^2-h^2(\theta-\theta_0)^2}$	$\pm\frac{c}{h}\tan\psi_1$	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\cosh^{-1}\left(\frac{h\mu-m^2r}{h\mu}\right)\right\}$	$\pm c\left\{\frac{k\cot\phi_1}{m^2}+\frac{h\mu}{m^3}\cosh^{-1}\left(\frac{h\mu-m^2r_1}{h\mu}\right)\right\}$	$s-s_0=\pm\left\{\frac{\mu}{m}\left(1-\frac{h^2}{m^2}\right)\cosh^{-1}\left(\frac{h\mu-m^2r}{h\mu}\right)-\frac{h\sqrt{R}}{m^2}\right\}$	$\pm\left\{\frac{\mu}{m}\left(1-\frac{h^2}{m^2}\right)\cosh^{-1}\left(\frac{h\mu-m^2r_1}{h\mu}\right)-\frac{h\sqrt{R}}{m^3}\right\}$	C D E
	b	$c^2=0$	$\frac{\mu}{h}(1\pm\cos\phi_1)$	$r=\frac{2\mu}{h[1-(\theta-\theta_0)^2]}$	$\pm\frac{k\cot\phi_1}{hr_1}$	$z-z_0=0$	0	$s-s_0=\mp\frac{\sqrt{R}}{h}$	$s_0=\pm\frac{k\cot\phi_1}{h}$	C D
III $h^2>c^2$ $\mu^2<k^2$	a	$c^2\neq0$	$\frac{h\mu+\sqrt{\Delta}}{m^2}$	$r=-\frac{n^2}{h\mu-\sqrt{\Delta}\cos\frac{n}{k}\theta}$	$\frac{k}{n}\cdot\frac{\pi}{2}$	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\cosh^{-1}\left(\frac{m^2r-h\mu}{\sqrt{\Delta}}\right)\right\}$	0	$s-s_0=\pm\left\{\frac{\mu}{m}\left(1-\frac{h^2}{m^2}\right)\cosh^{-1}\left(\frac{m^2r-h\mu}{\sqrt{\Delta}}\right)-\frac{h\sqrt{R}}{m^2}\right\}$	0	C D F
	b	$c^2=0$	$\frac{h\mu+\sqrt{\Delta}}{h^2}$	$r=\frac{-n^2}{h\mu-\sqrt{\Delta}\cos\frac{n}{k}\theta}$	$\frac{k}{n}\cdot\frac{\pi}{2}$	$z-z_0=0$	0	$s-s_0=\mp h\frac{\sqrt{R}}{m^2}$	0	C D F
IV [illegible]	[illegible]	$h^2\neq0$ [illegible]	$\frac{-h\mu+\sqrt{\Delta}}{m^2}$	$r=\frac{n^2}{h\mu+\sqrt{\Delta}\cosh\left[\pm\frac{n}{k}(\theta-\theta_0)\right]}$	0	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\sin^{-1}\left(\frac{m^2r+h\mu}{\sqrt{\Delta}}\right)\right\}$	$\mp\frac{ch\mu}{m^3}\cdot\frac{\pi}{2}$	$s-s_0=\pm\left\{\frac{\mu}{m}\left(1+\frac{h^2}{m^2}\right)\sin^{-1}\left(\frac{m^2r-h\mu}{\sqrt{\Delta}}\right)+\frac{h\sqrt{R}}{m^2}\right\}$	$\mp\frac{\mu}{m}\left(1+\frac{h^2}{m^2}\right)\frac{\pi}{2}$	A
	b	$h^2=0$ $k^2\neq0$	$\frac{n}{m}$	$r=\frac{n}{m\cosh\left[\pm\frac{n}{k}(\theta-\theta_0)\right]}$	0	$r^2+(z-z_0)^2=r_1^2$	0	$s-s_0=\pm\left\{\frac{\mu}{m}\sin^{-1}\frac{mr}{n}\right\}$	0	A
	c	$h^2\neq0$ $k^2=0$	$\frac{-h\mu+\sqrt{\Delta}}{m^2}$	$\theta-\theta_0=0$	0	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\sin^{-1}\left(\frac{m^2r+h\mu}{\sqrt{\Delta}}\right)\right\}$	$\mp\frac{ch\mu}{m^3}\cdot\frac{\pi}{2}$	$s-s_0=\pm\left\{\frac{\mu}{m}\left(1+\frac{h^2}{m^2}\right)\sin^{-1}\left(\frac{m^2r-h\mu}{\sqrt{\Delta}}\right)+\frac{h\sqrt{R}}{m^2}\right\}$	$\mp\frac{\mu}{m}\left(1+\frac{h^2}{m^2}\right)\frac{\pi}{2}$	A
	d	$h^2=0$ $k^2=0$	$\frac{\mu}{c}$	$\theta-\theta_0=0$	0	$\frac{r^2}{\mu^2}+\frac{(z-z_0)^2}{c^2\mu^2}=1$	0	$s-s_0=\pm\left\{\frac{\mu}{c}\sin^{-1}\frac{c}{\mu}r\right\}$	0	A
V $h^2<c^2$ $\mu^2=k^2$	a	$h^2\neq0$	$\frac{-2h\mu}{m^2}$	$r=\frac{-2h\mu}{h^2(\theta-\theta_0)^2+m^2}$	0	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\sin^{-1}\left(\frac{m^2r+h\mu}{\sqrt{\Delta}}\right)\right\}$	$\mp\frac{ch\mu}{m^3}\cdot\frac{\pi}{2}$	$s-s_0=\pm\left\{\frac{\mu}{m}\left(1+\frac{h^2}{m^2}\right)\sin^{-1}\left(\frac{m^2r+h\mu}{\sqrt{\Delta}}\right)+\frac{h\sqrt{R}}{m^2}\right\}$	$\mp\frac{\mu}{m}\left(1+\frac{h^2}{m^2}\right)\frac{\pi}{2}$	C
	b	$h^2=0$								G
VI [illegible]	a	$h^2\neq0$ [illegible]	[illegible]	$r=-\frac{n^2}{h\mu-\sqrt{\Delta}\cos\frac{n}{k}\theta}$	$\frac{k}{n}\cdot\frac{\pi}{2}$	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\sin^{-1}\left(\frac{m^2r-h\mu}{\sqrt{\Delta}}\right)\right\}$	$\mp\frac{ch\mu}{m^3}\cdot\frac{\pi}{2}$	$s-s_0=\pm\left\{\frac{\mu}{m}\left(1+\frac{h^2}{m^2}\right)\sin^{-1}\left(\frac{m^2r+h\mu}{\sqrt{\Delta}}\right)+\frac{h\sqrt{R}}{m^2}\right\}$	$\mp\frac{\mu}{m}\left(1+\frac{h^2}{m^2}\right)\frac{\pi}{2}$	C F
	b	$h^2=0$								G
VII $h^2=c^2$ $\mu^2>k^2$	a	$h^2\neq0$ $k^2\neq0$	$\frac{k\cot\phi_1}{c\tan\psi_1}$	$r=\frac{-n^2}{h\mu}\cdot\frac{1}{1\cosh\left[\pm\frac{n}{k}(\theta-\theta_0)\right]}$	$\pm\frac{k}{n}\log\frac{k\cot\phi_1-n}{k\cot\phi_1+n}$	$(z-z_0)^2=\frac{1}{9h^2\mu^4}\left[n^6-3h^2\mu^2n^2r^2-2h^3\mu^3r^3\right]$	$\mp\frac{(n^2+h\mu r_1)}{3\mu^2}r_1$	$s-s_0=\frac{\pm n^2+h\mu r-3\mu^2}{3h\mu^2}\sqrt{R}$	$\mp\frac{(2\mu^2-k^2-h\mu r_1)}{3\mu^2}r_1$	C E
	b	$h^2=0$ $k^2\neq0$	Z_1	$r=e^{\pm\frac{n}{k}(\theta-\theta_0)}$	0	$z-z_0=0$	0	$s-s_0=\frac{\pm(\mu r-hr^2)}{2}$	$\mp\left(\frac{\mu-h}{2}\right)$	C
	c	$h^2\neq0$ $k^2=0$	$\frac{-\mu}{h}$	$\theta-\theta_0=0$	0	$(z-z_0)^2=\frac{1}{9h^2\mu}\left(\mu^3-3h^2\mu r^2-2h^3r^3\right)$	0	$s-s_0=\frac{\pm n^2+h\mu r-3\mu^2}{3h\mu^2}\sqrt{R}$	$\mp\frac{(2\mu^2-h\mu r_1)}{3\mu^2}r_1$	C
VIII $h^2=c^2$ $\mu^2=k^2$	a	$h^2\neq0$	$\frac{k\cot\phi_1}{c\tan\psi_1}$	$r=\frac{-2k^2}{\mu h(\theta-\theta_0)^2}$	$\frac{-2\mu}{\sqrt{-2\mu hr_1}}$	$(z-z_0)^2=\frac{2}{9}\frac{c}{\mu}r^3$	$\mp\frac{\frac{2}{3}r_1^{\frac{3}{2}}h}{\sqrt{-2\mu h}}$	$(s-s_0)^2=\frac{-2r}{9h\mu}(3\mu-hr)^2$	$\mp\frac{2r_1^{\frac{1}{2}}(3\mu-hr_1)}{3\sqrt{-2\mu h}}$	C
	b	$h^2=0$	r_1	$r=r_1$		$z-z_0=0$	0			C H
IX $h^2=c^2$ $\mu^2<k^2$	a	$h^2\neq0$	$\frac{-n^2}{2h\mu}$	$r=\frac{-n^2}{\mu h}\frac{1}{1+\cos\frac{n}{k}(\theta-\theta_0)}$	0	$(z-z_0)^2=\frac{-1}{9\mu^4h^2}\left(2h^3\mu^3r^3+3h^2\mu^2n^2r^2-n^6\right)$	0	$s-s_0=\frac{\pm h\mu r-n^2-3\mu^2}{3h\mu^2}\sqrt{R}$	$\mp\frac{(2\mu^2-k^2-h\mu r_1)}{3\mu^2}r_1$	C F
	b	$h^2=0$								G

Notes on Preceding Table.

A. $h>0$.

B. $m<0$; $n>0$.

C. $h<0$.

D. $m>0$; $n<0$.

E. Since the value of r_1 used in Case I has become zero in this case, and since this contradicts the hypothesis that $F(r)$ is finite, it follows that $\phi_1 \neq \frac{1}{2}\pi$, $\psi_1 \neq 0$. The initial value of r cannot be a root of the equation $R=0$. When $r=r_1$, let $\phi=\phi_1 \neq \frac{1}{2}\pi$, $\psi=\psi_1 \neq 0$. With other initial conditions as before, we obtain for k the value

$$k=r_1 T_0 \frac{\tan \psi_1}{\cot \phi_1}.$$

F. In determining the equation of the cylinder for this case we have nsed the positive sign of equation (8) only. The negative sign gives a similar curve.

G. For $h=0$ the expression $\sqrt{R}$ becomes imaginary for all values of r.

H. To determine the curve of equilibrium for this case we have from equation (5), since $R=0$,

$$\frac{k}{r}\frac{dr}{d\theta}=0.$$

Since $k \neq 0$ and $r \neq 0$, we obtain on integration, $r=\text{constant}=r_1$.

Graphs of the guiding curves of the cylinder and a meridian section of the surface of revolution have been made for each case. Since there are so many cases to consider a reproduction of these graphs is not convenient here. In regard to the curve of equilibrium, the prevailing type is a spiral with the z-axis as the line about which it winds.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

357. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Solve the system

$$\sqrt{(x^2+a^2+b^2+c^2)}=\sqrt{(y^2+b^2+c^2)}+\sqrt{(z^2+b^2+c^2)},$$
$$\sqrt{(y^2+a^2+b^2+c^2)}=\sqrt{(x^2+a^2+c^2)}+\sqrt{(z^2+a^2+c^2)},$$
$$\sqrt{(z^2+a^2+b^2+c^2)}=\sqrt{(x^2+a^2+b^2)}+\sqrt{(y^2+a^2+b^2)}.$$

Solution by B. F. FINKEL, Ph. D., Drury College.

Transposing the first term of the second member of the first equation, squaring, collecting, and transposing, we have

$$x^2+y^2-z^2+s^2=2\sqrt{(x^2+s^2)}\sqrt{(y^2+s^2-a^2)}\ldots(1),$$

where $s^2=a^2+b^2+c^2$. Similarly, we obtain from the second equation,

$$x^2+y^2-z^2+s^2=2\sqrt{(y^2+s^2)}\sqrt{(x^2+s^2-b^2)}\ldots(2),$$

and from the third,

$$x^2+z^2-y^2+s^2=2\sqrt{(z^2+s^2)}\sqrt{(x^2+s^2-c^2)}\ldots(3).$$

By transposing the second term of the first equation, squaring, collecting, and retransposing, we get

$$x^2+z^2-y^2+s^2=2\sqrt{(x^2+s^2)}\sqrt{(z^2+s^2-a^2)}\ldots(4).$$

From (1) and (2) we have,

$$\sqrt{(x^2+s^2)}\sqrt{(y^2+s^2-a^2)}=\sqrt{(y^2+s^2}\sqrt{(x^2+s^2-b^2)};$$

whence, by squaring and dividing by $(x^2+s^2)(y^2+s^2)$, we get

$$\frac{y^2+s^2-a^2}{y^2+s^2}=\frac{x^2+s^2-b^2}{x^2+s^2}.$$

Hence, $\dfrac{a^2}{y^2+s^2}=\dfrac{b^2}{x^2+s^2}$, or $y^2+s^2=\dfrac{a^2}{b^2}(x^2+s^2)$.

From (3) and (4), we obtain, in like manner,

$$z^2+s^2=\frac{a^2}{c^2}(x^2+s^2).$$

Substituting these values of z^2+s^2 and y^2+s^2 in the first equation, we have

$$\sqrt{(x^2+s^2)}=\sqrt{\frac{a^2}{b^2}(x^2+s^2)-a^2}+\sqrt{\frac{a^2}{c^2}(x^2+s^2)-a^2};$$

whence, $bc\sqrt{(x^2+s^2)}=ab\sqrt{(x^2+s^2-b^2)}+ab\sqrt{(x^2+s^2-c^2)}$.

Squaring,

$$b^2c^2(x^2+s^2)=a^2c^2(x^2+s^2-b^2)+2a^2bc\sqrt{(x^2+s^2-b^2)}\sqrt{(x^2+s^2-c^2)}+a^2b^2(x^2+s^2-c^2);$$

transposing, and combining,

$$(b^2c^2-a^2c^2-a^2b^2)(x^2+s^2)+2a^2b^2c^2=2a^2b^2c^2\sqrt{(x^2+s^2-b^2)}\sqrt{(x^2+s^2-c^2)}.$$

Squaring both members of this equation, and rearranging terms, we have

$$[(b^2c^2-a^2c^2-a^2b^2)^2-4a^4b^2c^2](x^2+s^2)+4a^2b^4c^4(x^2+s^2)=0.$$

$\therefore x^2+s^2=0$, or $x^2+s^2=$

$$\frac{4a^2b^4c^4}{4a^4b^2c^2-(b^2c^2-a^2c^2-a^2b^2)^2}$$

$$=\frac{4a^2b^4c^4}{(b^2c^2-a^2c^2-a^2b^2+2a^2bc)(-b^2c^2+a^2c^2+a^2b^2+2a^2bc)}$$

$$=\frac{4a^2b^4c^4}{[b^2c^2-a^2(b-c)^2][a^2(b+c)^2-b^2c^2)]}$$

$$=\frac{4a^2b^4c^4}{(bc+ab-ac)(bc-ab+ac)(ab+ac+bc)(ab+ac-bc)}$$

$=\dfrac{4a^2b^4c^4}{\triangle}$, where $\triangle$ is the denominator of the above fraction.

$$y^2+s^2=\frac{4a^4b^2c^4}{\triangle}, \text{ and } z^2+s^2=\frac{4a^4b^4c^2}{\triangle}.$$

$$\therefore x=\pm\left(\frac{4a^4b^2c^4-s^2\triangle}{\triangle}\right)^{\frac{1}{2}}, \ y=\pm\left(\frac{4a^4b^2c^4-s^2\triangle}{\triangle}\right)^{\frac{1}{2}}, \text{ and}$$

$$z=\pm\left(\frac{4a^4b^4c^2-s^2\triangle}{\triangle}\right)^{\frac{1}{2}}.$$

$x^2+s^2=0$ is not admissible.

PROBLEMS FOR SOLUTION.

ALGEBRA.

363. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

(a) If a and n be positive integers, the integral part of $[a+\sqrt{(a^2-1)}]^n$ is odd.

(b) If a and n be positive integers, the integral part of $[\sqrt{(a^2+1)}+a]^n$ is odd when n is even and even when n is odd. [From Todhunter's *Algebra*, p. 353].

364. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The English physicist, Hooke, published the discovery contained in the Latin sentence, "Ut tensio sic vis" by the cypher *cciiinossssttuv*. Preserving the lexicographical order, find which permutation, taking all letters, the Latin sentence is from the cypher.

365. Proposed by C. N. SCHMALL, New York City.

In still water, a steam tug goes 6 miles an hour less when towing a barge than when alone. Having drawn the barge 30 miles up a stream, whose current runs 1 mile an hour, it returns alone and completes the journey in 12 8/11 hours. Find the rate of the tug in still water.

GEOMETRY.

396. Proposed by DANIEL KRETH, Oxford, Iowa.

In the triangle ABC, $AB=214$, $BC=263$, and $AC=405$. A point P is situated in the same horizontal plane; angle $BPA=13°\ 30'$ and angle $BPC=29°\ 50'$. Find the distances, AP, BP, and CP.

397. **Proposed by DAVID F. KELLEY, New York City.**

If ABC be a semicircle and CD a perpendicular from C on the diameter AB, prove that the radius of the circle inscribed in the triangle ABC equals half the sum of the radius of the circle touching arc AC and the sides AD and DC of the triangle ADC, and the radius of the circle touching arc CB and sides DB and DC of triangle CDB, and that the centers of the three circles are collinear.

398. **Proposed by C. N. SCHMALL, New York City.**

In a square $ABCD$ draw the diagonal AC. Now bisect AD in G and draw GB cutting AC in H. Prove that $\triangle AGH=\frac{1}{2}\triangle CGH=\frac{1}{3}\triangle ABG=\frac{1}{4}\triangle BCH$.

CALCULUS.

318. **Proposed by JOHN C. GREGG, Greencastle, Ind.**

A thread is wound spirally n times around a cone, the radius of whose base is r, and slant height h, the turns being at uniform distance apart. If the thread is kept taut, what will be the length of the trace of its end on a horizontal plane?

319. **Proposed by C. N. SCHMALL, New York City.**

Given $u=\frac{yz}{x}$, $v=\frac{zx}{y}$, $w=\frac{xy}{z}$, prove

$$\begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y}, & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y}, & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x}, & \frac{\partial w}{\partial y}, & \frac{\partial w}{\partial z} \end{vmatrix}=4xyz.$$

MECHANICS.

265. **Proposed by A. H. HOLMES, Brunswick, Maine.**

A gun is mounted in a fort at height h above the sea, and a similar gun is mounted on a ship. Show that there is a region of area $4\pi rh$ within which the ship is within range of the fort while the fort is out of range of the ship, r being the maximum range of either gun on a horizontal plane through it.

266. **Proposed by A. M. HARDING, Assistant Professor of Mathematics, University of Arkansas.**

A, B, C are three equidistant smooth pegs in the same horizontal line, and a heavy uniform string has its ends tied to A, C, and is looped over B. Show that there may or may not be a position of equilibrium in which the two catenaries, AB, BC, are unequal, and if there is such a position it will be stable. Show that the position of equilibrium in which the middle point of the string is at B is unstable or stable according as an unsymmetrical position of equilibrium does or does not exist. [Jeans' *Mechanics*, page 187].

AVERAGE AND PROBABILITY.

208. **Proposed by A. M. HARDING, Assistant Professor of Mathematics, University of Arkansas.**

Find the chance that the distance of two points within a square shall not exceed a side of the square.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

185. Proposed by R. D. CARMICHAEL, Bloomington, Ind.

Obtain the complete solution of the equatian $\phi(p^\alpha)=\phi(q^\beta)$ where ϕ denotes Euler's ϕ-function, p and q are unknown primes and α and β are unknown integers.

NOTES AND NEWS.

There has recently appeared from the publishers, Gauthier-Villars, a fine translation, by Paul Babarin, of Dr. Halsted's *Rational Geometry*. The book since its publication in English a few years ago, has been translated in German and Japanese, and now in French. In the preface of the French edition, C. A. Laisant says, among other things, that the book is clearly written, and the translation faithful and limpid, and he expresses the belief that the work is destined to exercise a profound effect upon the transformation of geometric teaching. F.

The reports of the American Committee of the International Commission on the Teaching of Mathematics, which are now being issued by the United States Bureau of Education, may be had upon application to the Commissioner of Education, Washington, D. C. Those who desire to receive these reports may thus not only get the half dozen already issued but may have their names entered so as to receive the remaining reports as fast as they are issued.

The final report of the International Commissioners is to be presented at the meeting of the International Congress of Mathematicians to be held in Cambridge, England, in August, 1912. S.

The annual resumé of doctorates conferred by American Universities, as printed in *Science* in the issue of August 18, 1911, should be of interest to readers of the MONTHLY.

It contains comparative tables showing the average number of doctorates in all subjects for each of 44 universities for the ten years, 1898-1907; the total number for the fourteen years, 1898-1911; the actual number for each of the years, 1908, 1909, 1910, 1911, together with the correspending data for doctorates in the sciences including mathematics.

The universities which have conferred more than 100 doctorates during the last fourteen years are: Columbia 555, Chicago 545, Harvard 495, Yale 452, Johns Hopkins 411, Pennsylvania 341, Cornell 306, Wisconsin 152, Clark 137, and New York 123. Those showing an average of more than 10 during the ten years 1898-1907 are: Chicago 35.6, Harvard 33.8, Columbia 32.2, Yale 31.8, Johns Hopkins 30,5, Pennsylvania 22.5, and Cornell 18.1.

Those conferring more than 20 doctorates in 1911 are: Columbia 75, Chicago 55, Harvard 42, Cornell 34, Yale 31, Pennsylvania 29, Johns Hopkins 28.

In the sciences the institutions conferring a total of more than 100 doctorates in the last fourteen years are: Chicago 280, Johns Hopkins 239, Columbia 218, Harvard 198, Cornell 197, Yale 194, Pennsylvania 143, and Clark 126. Those having an average of more than 10 during the ten years, 1898-1907, are: Johns Hopkins 16.8, Chicago, 16.4, Harvard 14.1, Columbia 13.4, Yale 12.4, and Cornell 10.4. Those conferring 10 or more doctorates in the sciences in 1911 are: Chicago 35, Columbia 29, Cornell 27, Harvard 20, Johns Hopkins 19, Clark 16, Yale 15, Wisconsin 13, and Pennsylvania 10.

The total number of doctorates conferred in the last fourteen years by the 44 universities given in the tables is 4286, the total number in the sciences being 2037, or 48 per cent of the whole number. The total number for 1911 was 437, of which 239 were in the sciences, and the total average for the ten years, 1898-1907, was 272.4 of which 124.1 belong to the sciences.

The numbers in the various departments are also of interest. Those departments credited with a total of over 100 during the last fourteen years are chemistry 533, physics 264, zoölogy 244, psychology 222, mathematics 206, botany 183, English 121, geology 114, and history 105. Those departments showing more than 10 doctorates in 1911 are: chemistry 65, physics 37, English 33, history 26, philosophy 26, zoölogy 25, mathematics 25, psychology 23, education 23, botany 20, sociology 18, economics 16, geology 15, romance languages 12, Latin 11, and agriculture 11.

The following quotation from the report is significant of the trend with respect to the sciences:

"The number of doctorates in the natural and exact sciences is increasing more rapidly than in other subjects. Prior to 1908 the average number of degrees conferred in the sciences was 124, as compared with 198 in the other group, in the three following years the average numbers were 186 and 189, respectively; and this year the numbers were 239 and 198. As shown in the table, Chicago is the university which has conferred the largest number of degrees in the natural and exact sciences, followed by Johns Hopkins and Columbia. Of the degrees conferred by Cornell, 64 per cent have been in the sciences, at the Johns Hopkins 58 per cent, at Harvard 40 per cent, at Columbia 37 per cent. It is somewhat curious that the percentage at Wisconsin, Michigan, Illinois, and Minnesota should be as small as 42, 39, 54, and 37, respectively, as it is the general impression that the sciences are especially emphasized at the State universities."

The 25 doctorates in mathematics in 1911 were distributed as follows: Chicago 5, Yale 5, Clark 2, Johns Hopkins 2, Princeton 2, Pennsylvania 2, Columbia 1, Illinois 1, Michigan 1, Syracuse 1, Harvard 1, Cornell 1, and Cincinnati 1.

S.

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BOOKS AND PERIODICALS.

An Introduction to Thermodynamics. By John Mills, Professor of Physics and Electrical Engineering, Colorado College, Colorado Springs. 8vo. Cloth, viii+136 pages. Price, $2.00. Boston: Ginn & Co.

"This book is a clearly and carefully written presentation of the portions of elementary thermodynamics which are essential to the present-day engineer. It is suitable for use in third-year engineering courses or in regular college courses in thermodynamics. The text constitutes at the same time a complete course for general scientific or engineering students and an adequate preparation for further work in pure thermodynamics or in the special study of the design and operation of steam engines, turbines, heat engines, or compressed-air machines.

Although it makes use of elementary calculus it does not necessarily require a knowledge of calculus further than the fundamental concepts of a derivative and of an integration. It is intended to follow a good college course in general physics, but it includes a brief resume of such portions of this subject as are essential to the development of the subject of thermodynamics.

The text is illustrated by a careful selection of diagrams and plots. Illustrative problems cover the entire range of the material presented in the text and are accompanied by numerical solutions. These examples and the additional problems for solution are peculiarly strong and attractive features of the book."

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